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Euler Integrals for Commuting SLEs

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Schramm-Loewner Evolutions (SLEs) have proved an efficient way to describe a single continuous random conformally invariant interface in a simply-connected planar domain; the admissible probability distributions are parameterized by a single positive parameter κ. As shown in, Ref. 8 the coexistence of n interfaces in such a domain implies algebraic ("commutation") conditions. In the most interesting situations, the admissible laws on systems of n interfaces are parameterized by κ and the solution of a particular (finite rank) holonomic system.

The study of solutions of differential systems, in particular their global behaviour, often involves the use of integral representations. In the present article, we provide Euler integral representations for solutions of holonomic systems arising from SLE commutation. Applications to critical percolation (general crossing formulae), Loop-Erased Random Walks (direct derivation of Fomin's formulae in the scaling limit), and Uniform Spanning Trees are discussed. The connection with conformal restriction and Poissonized non-intersection for chordal SLEs is also studied.

KEY WORDS: correlation functions, critical percolation, Schramm-loewner evolutions.

1. INTRODUCTION

Critical systems in the plane (such as critical percolation, self-avoiding walk, . . .) are generally conjectured to have scaling limits satisfying certain conformal invariance properties. Schramm-Loewner Evolutions (SLE), introduced in Ref. 25, have proved a powerful tool to describe and study these limits.

In several of these discrete models, such as percolation, one can define more than one interface. Scaling limits of these systems of interfaces should have similar conformal invariance properties. If one studies the limit of a single simple path,

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then conformal invariance and a "Markov" property specify the limit, if it exists, up to a positive parameter, denoted by κ; this limit is SLE_κ.

If one considers the joint law of several random curves, then the limit can be encoded through Loewner's equations as (one-dimensional) diffusions. The coefficients now depend on several parameters describing the boundary conditions. But these coefficients have to satisfy some compatibility conditions, expressed as commutation relations for associated infinitesimal generators. These commutation conditions can be recast as systems of linear PDEs with meromorphic coefficients and regular singularities along hyperplanes.[8] The key idea here is that we want to define distributions on systems of geometric paths, independently of any (artificial) time parameterization.

A particularly interesting set-up is the following: $(2n)$ points are marked on the boundary of a planar simply-connected domain, and these $(2n)$ points are joined by n curves. Then it is shown in Ref. 8 that the possible systems of random curves can be obtained by solving a certain holonomic system, which is parameterized by κ and n.

In this paper, we shall be mainly interested in the properties of this family of holonomic systems introduced in Ref. 8. Solutions of this system can be related to particular (local) martingales of SLE. These local martingales are essentially the Girsanov densities for the systems of random curves w.r.t. independent SLEs. Our main goal here is to give integral representations of these solutions. When studying global properties of solutions of differential equations (asymptotics/boundary conditions, monodromy, ...), use of such integral representations is crucial. Also, although the theory of holonomic systems is rich and deep, getting solutions of a particular system is often challenging.

Cardy's formula for critical percolation[3,26] gives the probability that there exists a crossing from left to right in a rectangle. For $\kappa = 6$, the system generalizes Cardy's formula to situations were sides of a $(2n)$-gon are set to alternate colors. Other related formulae have been proposed or proved, such as a formula for expected number of crossings,[4] for crossing of annuli,[5] Watts' formula for rectangles,[9,27] and Pinson's formula for elliptic curves.[22] These formulae depend on one parameter (which is complex in Pinson's formula).

For $\kappa \in (0, 8/3)$, the solutions of the system can be interpreted in the continuous limit in terms of nonintersection of n independent SLE_κ and independent "loop-soups"[16,21,29] with appropriate intensity. If $\kappa = 8/3$, this loop-soup is empty, and we are studying the probability that n independent $\mathrm{SLE}_{8/3}$ do not intersect. As $\mathrm{SLE}_{8/3}$ is conjectured to be the scaling limit of Self-Avoiding Walks,[19] one can think in terms of the scaling limit of non-intersecting self-avoiding walks.

In the case $\kappa = 2$, one can see that the solutions of the system are exceptionally rational and connect them to Fomin's determinantal formulae for Loop-Erased Random Walks (see Refs. 12, 15, 20). This can also be seen as a particular case of the previous restriction construction (in the continuum). In the case

$\kappa = 8$, corresponding to the scaling limit of the Uniform Spanning Tree (UST, see Ref. 20), one can study the situation where alternate wired and free conditions on the boundary of a domain force the creation of n non-intersecting Peano paths on the Manhattan lattice. The scaling limit can be identified through Wilson's algorithm.

It has been brought to our attention that the solutions obtained here appear in the so-called Coulomb Gas formalism (see Ref. 7, Chapter 9 in Ref. 6, and references therein), where they are referred to as Coulomb Gas representations. In this framework, deriving integral representations for correlation functions involves (chiral) vertex operators and screening operators, and the Feigin-Fuchs integral representation. Further developments along these lines pertain to monodromy, fusion, and BRST cohomology.

In the present article, we obtain directly these integral representations starting from the associated holonomic system. The analysis of this system leads for example to results on rank and reductions using symmetry (as in Sec. 4.4). Also, we focus on the interplay between the algebraic, analytic and probabilistic aspects of the problem. For instance, delicate analytic questions regarding boundary values can be bypassed by probabilistic arguments (Sec. 6). In the case of Loop-Erased Random Walks and Uniform Spanning Trees ($c = -2$), direct connections with the combinatorial models are also established (3.2, 3.3).

The set-up, objects and notations are essentially the same as in Ref. 8 and are mainly the standard SLE notations ((g_t), $\kappa, \ldots$), so we will recall them only briefly. This article can be read either as a self-contained study of a particular family of holonomic systems, or as a companion paper to Ref. 8.

The paper is organized as follows. After some background and notations, several situations leading to natural systems of n curves are described. Discrete models, such as critical percolation, loop-erased random walks (in relation with Fomin's formulae), and uniform spanning trees (in relation with Wilson's algorithm) provide important examples. In the continuous set-up, the theory of conformal restriction and loop-soups[(16,21)] can also be connected to the problem of commuting SLEs.[(8)] In the following section, as a preparation, several particular cases (particular values of κ and n) are briefly studied; in particular, for $\kappa = 6$ and $2n = 6$, we get new crossing probability expressions for critical percolation in conformal hexagons. Building on these particular solutions, general (formal) solutions are given, as Euler integral representations. We then discuss the problem of identifying solutions corresponding to specific geometric configurations; or, in other terms, how to choose the cycle of integration in the Euler integral.

2. BACKGROUND AND NOTATIONS

First we recall some definitions and fix notations. We shall be mainly interested in chordal versions of SLE, i.e. random curves connecting two boundary points of a plane simply-connected domain, satisfying some conformal invariance properties. For general background on SLE, see Refs. 18, 23, 28. Also, we will use at some point results on the restriction property and the "loop-soup" (see Refs. 16, 21, 29).

Consider the family of ODEs, indexed by z in the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$$

with initial conditions $g_0(z) = z$, where W_t is some real-valued (continuous) function. These chordal Loewner equations are defined up to explosion time τ_z (maybe infinite). Define: $K_t = \overline{\{z \in \mathbb{H} : \tau_z < t\}}$. Then $(K_t)_{t\geq 0}$ is an increasing family of compact subsets of $\overline{\mathbb{H}}$; moreover, g_t is the unique conformal equivalence $\mathbb{H}\backslash K_t \to \mathbb{H}$ such that (hydrodynamic normalization at ∞):

$$g_t(z) = z + o(1).$$

For any compact subset K of $\overline{\mathbb{H}}$ such that $\mathbb{H}\backslash K$ is simply-connected, we denote by ϕ the unique conformal equivalence $\mathbb{H} \to \mathbb{H}\backslash K$ with hydrodynamic normalization at ∞; so that $g_t = \phi_{K_t}$. The coefficient of $1/z$ in the Laurent expansion of g_t at ∞ is by definition the half-plane capacity of K_t at infinity; this capacity equals $(2t)$.

If $W_t = x + \sqrt{\kappa} B_t$ where (B_t) is a standard Brownian Motion, then the Loewner chain (K_t) (or the family (g_t)) defines the chordal Schramm-Loewner Evolution with parameter κ in $(\mathbb{H}, x, \infty)$. The chain K_t is generated by the trace γ, a continuous process taking values in $\overline{\mathbb{H}}$, in the following sense: $\mathbb{H}\backslash K_t$ is the unbounded connected component of $\mathbb{H}\backslash\gamma_{[0,t]}$.

The trace is a continuous non self-traversing curve. It is a.s. simple if $\kappa \leq 4$ and a.s. space-filling if $\kappa \geq 8$. Its Hausdorff dimension is a.s. $1 + \kappa/8$ if $\kappa \leq 8$ (and 2 otherwise).

If (D, x, y) is a simply-connected domain with two marked points on the boundary, SLE_κ from x to y in D is defined as the image of SLE_κ from 0 to ∞ in $\mathbb{H}$ (as defined above) by a conformal equivalence $(\mathbb{H}, 0, \infty) \to (D, x, y)$ (that exists by Riemann's mapping theorem). With this definition, SLE satisfies a "Domain Markov" property, which, together with conformal equivalence, essentially characterizes it.

In Ref. 8, the question of defining several SLE strands simultaneously in a domain is addressed (say each of those strands is absolutely continuous w.r.t. SLE_κ). The key point is that an appropriate "Domain Markov" condition for the joint law imposes stark "commutation conditions" on the drift terms of the driving processes. Elucidating those conditions is the main object of Ref. 8. We summarize the result in the following case: $(2n)$ points $x_1, \ldots, x_{2n}$ are marked on the boundary of $\mathbb{H}$, and we want to define jointly n SLEs connecting these $(2n)$ points, such that the joint distribution depends only on the position of the marked points, is Möbius invariant and satisfies the appropriate Markov property. Then necessarily, the SLE growing at x_i is driven by:

$$dX_t^{(i)} = \sqrt{\kappa} dB_t^{(i)} + \kappa \frac{\partial_i \varphi}{\varphi}(g_t(x_1), \dots, X_t^{(i)}, \dots g_t(x_{2n}))\, dt$$

where $B^{(i)}$ is a standard Brownian motion and φ is a non-vanishing function annihilated by the operators:

$$\begin{cases} \frac{\kappa}{2}\partial_{kk} + \sum_{l \neq k} \frac{2\partial_l}{x_l - x_k} + \frac{\kappa - 6}{\kappa} \sum_{l \neq k} \frac{1}{(x_l - x_k)^2}, \qquad k = 1, \dots, 2n \\ \Sigma_k \partial_k \\ \Sigma_k x_k \partial_k - n(1 - 6/\kappa) \\ \Sigma_k x_k^2 \partial_k - (1 - 6/\kappa)(x_1 + \dots + x_{2n}) \end{cases} \tag{2.1}$$

From a CFT point of view, this corresponds to differential equations for correlation functions of $\phi_{1;2}$ primary fields.

If $n = 1$, $\varphi(x_1, x_2) = (x_2 - x_1)^{1-6/\kappa}$, and:

$$dX_t^{(1)} = \sqrt{\kappa} dB_t^{(1)} + \frac{\kappa - 6}{X_t^{(1)} - g_t(x_2)}\, dt$$

which is ordinary chordal SLE_κ from x_1 to x_2 (with a homographic change of coordinate).

The main question addressed in this article is how to obtain general integral representations for solutions of this system and their interpretation in terms of scaling limits for different discrete models.

3. PROBABILISTIC INTERPRETATIONS

In this section, we discuss probabilistic situations giving rise to natural examples of systems of geometric paths. We start with critical percolation, and then consider two other discrete models with particular harmonic properties, the Loop-Erased Random Walk (LERW) and the Uniform Spanning Tree (UST). We then proceed to show how to use the Poissonian structure of the loop-soup to define a natural notion of non-intersection for n chordal SLE_κ's, where $0 < \kappa \leq 8/3$.

3.1. Crossing Events for Critical Percolation

We discuss here consequences of the locality property for SLE_6/critical percolation, in particular regarding holonomic systems.

First, we describe a family of percolation events that appear as particularly well-suited for an SLE analysis. Recall the set-up of Cardy's formula, relating to critical percolation in a conformal quadrilateral. For simplicity, we will consider site percolation on the triangular lattice: each site is colored in blue or yellow with probability 1/2, all sites being independent. (Alternatively, the hexagons of a honeycomb tiling are colored in blue, yellow). A subgraph of the triangular lattice, with mesh $\varepsilon \searrow 0$, approximates a quadrilateral. As boundary conditions, a pair of opposite edges of the quadrilateral is set to blue, while the other pair is set to yellow. For (plane) topological reasons, either the two yellow edges are connected by a yellow path on the lattice, or the two blue edges are connected by a blue path (see Fig. 1).

It seems quite natural to generalize this construction to $(2n)$-gons, $n \geq 2$. So consider a bounded Jordan domain $D \subset \mathbb{C}$; $a_1 \dots a_{2n}$ are distinct points on the boundary (in counterclockwise order, say). The boundary conditions are alternate: for $1 \leq i \leq n$, the are (a_{2i}, a_{2i+1}) is set to yellow, and (a_{2i-1}, a_{2i}) is set to blue (with cyclical indexing, i.e. $a_{2n+1} = a_1$). We are interested in the connectivity properties of the random percolation graph that are observable from the boundary. Denote by e_i the edge (a_i, a_{i+1}). Let $c(e_i, e_j) = 1$ if e_i, e_j are of the same color and are connected by a path of this color, and $c(e_i, e_j) = 0$ otherwise. Note that a path can include a portion of the boundary. An elementary event is an event of type

$$C(\epsilon) = \bigcap_{1 \leq i < j \leq 2n} \{c(e_i, e_j) = \epsilon_{i,j}\}$$

where $\epsilon = (\epsilon_{i,j})_{1 \leq i < j \leq 2n}, \epsilon_{i,j} \in \{0, 1\}$. We now describe the non-empty elementary events. We suppose that the mesh ε is small enough, so that the edges are

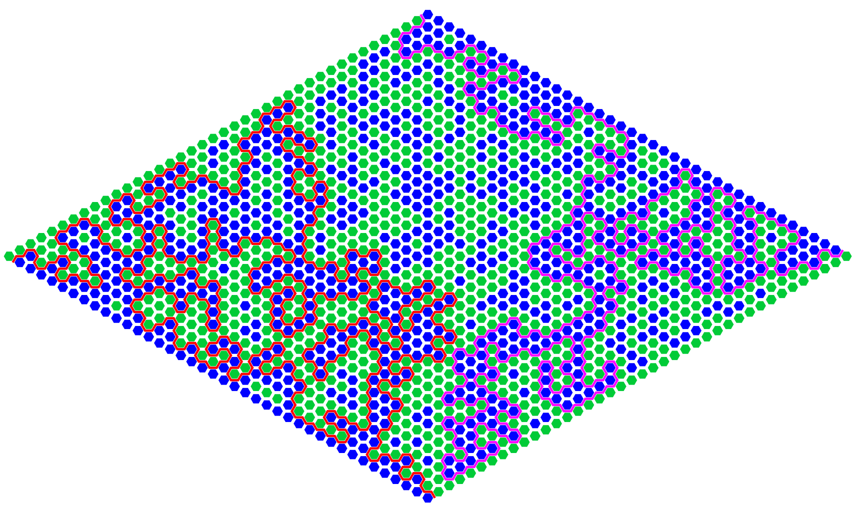

Fig. 1. Two interfaces (red, fuchsia) bound a crossing of the lozenge.

nonempty and disjoint. From the Russo-Seymour-Welsh theory (and FKG inequality, see Ref. 14), the elementary events that are topologically possible will happen with probability bounded away from 0 (and from 1) as the mesh ϵ goes to zero.

Let $n \geq 2$. There are exactly C_n (non-empty) elementary events with probability bounded away from 0 as $\varepsilon \searrow 0$, where C_n denotes the n-th Catalan's number:

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

Indeed, it is easy to see that there is a one-to-one correspondence between elementary events and non-crossing partitions of the set of blue edges. As is well known, this number is C_n. This can be seen as follows: consider the smallest $i > 1$ such that (a_{2i-1}, a_{2i}) is connected to (a_1, a_2) by a blue path; this leads directly to the recurrence relation for Catalan's numbers. For instance, for $n = 3$, one gets $C_3 = 5$ possible configurations for a conformal hexagon with alternate boundary conditions, as illustrated by Fig. 2.

Alternatively, at each point a_i where boundary conditions change, one can start an exploration path[25,26] that winds between the connected components of (a_{i-1}, a_i) and (a_i, a_{i+1}). Such a path is simple and ends at a_j, for some j (i and j have opposite parity). So the ($2n$) boundary points are paired by n non-intersecting exploration processes. This pairing is random; the number of such pairings (satisfying the non-crossing condition) is C_n. Note also that a site that touches two exploration processes is pivotal for these events. For bond percolation on $\mathbb{Z}^2$, it is convenient to represent the interfaces between connected components in $\mathbb{Z}^2$ and connected components for the associated percolation configuration on the dual graph. The collection of these interfaces (with appropriate boundary conditions) is made of closed loops and n non-intersecting simple paths connecting the ($2n$) "free" points on the boundary (see Fig. 3).

The exploration process for critical site percolation on the triangular lattice converges to SLE_6 in the scaling limit.[2,26] It is conjectured, and supported by numerical evidence, that it is also the case for critical bond percolation on $\mathbb{Z}^2$ (and also for more general lattices).

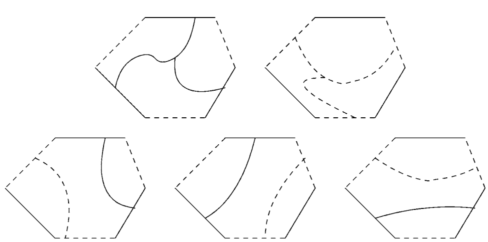

Fig. 2. The five configurations for a conformal hexagon (schematic).

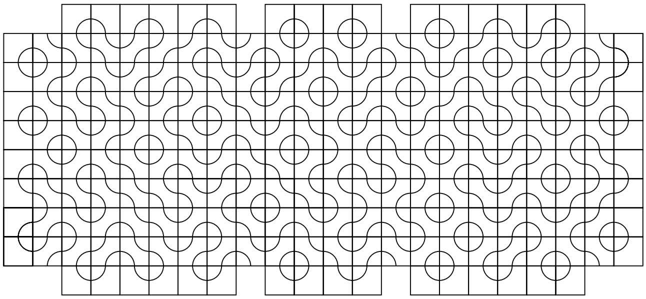

Fig. 3. Four paths in bond percolation on $\mathbb{Z}^2$.

From conformal invariance of the scaling limit of critical percolation, the probability of any of these elementary events should define a function on the corresponding moduli space (i.e. the space of Jordan domains with $(2n)$ (distinct) marked boundary points modulo conformal equivalence). Denote by $\mathcal{M}_{2n}$ this moduli space, which can be seen as a smooth $(2n-3)$-dimensional manifold, for $n \geq 2$ (for a discussion of SLE and moduli space, see e.g., Ref. 13). Considering all the possible configurations, one defines a function $\mathcal{M}_{2n} \to \mathbb{R}^{C_n}$. Assuming that this function is smooth, the dimension of the image is at most $(2n-3)$, which implies the existence of a large number of smooth relations between probabilities of elementary events. Indeed, the dimension of the moduli space, $(2n-3)$, is negligible compared with C_n:

$$C_n = \frac{\binom{2n}{n}}{n+1} \sim \frac{1}{n} \cdot \frac{\sqrt{2\pi 2n}(2n)^{2n} e^{-2n}}{(\sqrt{2\pi n}(n^n)e^{-n})^2} \sim \sqrt{\frac{2}{\pi}} \cdot \frac{4^n}{n^{3/2}}$$

as follows from Stirling's formula. The sum of all probabilities is 1; the nature of other smooth relations between these probabilities is unclear. Letting a given edge of the conformal $(2n)$-gon shrink to 0, one gets a conformal $(2n-2)$-gon, so that the corresponding part of the boundary of $\mathcal{M}_{2n}$ can be identified with $\mathcal{M}_{2n-2}$. Considering this operation for any edge, it appears that any affine relation between the C_n probabilities is proportional to the trivial (normalization) relation.

Assuming conformal invariance, one can give a differential characterization of these probabilities. More specifically, let $f(x_1, \ldots, x_{2n})$ be the probability (in the scaling limit) of any one of the C_n elementary events associated with the configuration $(\mathbb{H}, x_1, \ldots, x_{2n})$, $x_1 < \cdots < x_{2n}$. Let $i \in \{1 \ldots 2n\}$; we consider an infinitesimal percolation hull at x_i. The boundary changes color at x_i; Smirnov's key result (see Ref. 26) is that the percolation exploration process started from x_i converges to SLE_6 started at x_i (be it chordal SLE_6 to another boundary point, or radial SLE_6 to an inner point, since all these are equivalent for short enough times). Let $(\gamma_u)_{0\leq u\leq\eta}$ be the exploration process, and $(K_u)_{0\leq u\leq\eta}$ be the associated family of hulls, where η is such that the vertices x_j, $j\neq i$, are not disconnected at time η. Then the elementary event holds for $(\mathbb{H}, x_1, \dots, x_{2n})$ if and only if it holds for the (random) conformal $(2n)$-gon $(\mathbb{H}\backslash K_u, x_1, \dots, \gamma_u, \dots x_{2n})$, for any $u\in[0,\eta]$; assuming conformal invariance, this has probability $f(g_u(x_1), \dots, W_u, \dots g_u(x_{2n}))$, where (g_u) are the conformal equivalences defining the SLE$_6$ process. So $(f(g_u(x_1), \dots, W_u, \dots g_u(x_{2n})))_{0\leq u\leq\eta}$ is a martingale. To sum up, assuming that the probability of an elementary event defines a smooth function on $\mathcal{M}_{2n}$, then the function f is annihilated by the following differential ideal:

$$I_n = \langle \mathcal{L}_1, \dots \mathcal{L}_{2n}, \ell_{-1}, \ell_0, \ell_1\rangle$$

where $\mathcal{L}_i = 3\partial_{ii} + \sum_{j\neq i}\frac{2}{x_j-x_i}\partial_j$ (infinitesimal generator for SLE$_6$ growing at x_i), and $\ell_k = -\sum x_i^{1+k}\partial_i$. Note that the (real) Lie algebra generated by the $(\ell_k)_{k\in\mathbb{Z}}$ is (isomorphic to the) Witt algebra, and that the subalgebra $\langle\ell_{-1}, \ell_0, \ell_1\rangle$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, the tangent algebra of the Moebius group (the group of conformal automorphisms of a simply connected domain). The fact that one can explore the $(2n)$-gon starting at any of its vertices is a feature of locality.

It is readily seen that I_n, which is an ideal of differential operators with rational coefficients, is holonomic, i.e. the vector space of functions annihilated by I_n in the neighbourhood of a generic point has finite dimension (the rank of I_n is the dimension of this space). For background on holonomic systems, see e.g. Ref. 31. It is elementary that the rank of I_n is no greater that 4^n. Indeed, if f, defined in a neighbourhood U of a generic point, is annihilated by I_n, let $F : U \to \mathbb{R}^{\{0,1\}^{2n}}$, where:

$$F = \left(\partial_1^{\epsilon_1}\dots\partial_{2n}^{\epsilon_{2n}} f\right)\epsilon \in \{0,1\}^{2n}.$$

Using the operators $\mathcal{L}_i$, it readily follows that one can write $\partial_i F = M_i F$ for $i = 1\dots 2n$, where M_i is a matrix of rational coefficients. A local solution of this system in a neighbourhood of $(x_1, \dots, x_{2n})$ is entirely determined by the initial condition $F(x_1, \dots, x_{2n})$, so the solution space is of dimension at most $2^{2n} = 4^n$ (it would be exactly of dimension 4^n if the Frobenius integrability conditions were satisfied: $\partial_j M_i - \partial_i M_j = M_j M_i - M_i M_j$). Since $F_{(0,\dots,0)} = f$, the rank of I_n is at most 4^n. Note that we have not used the operators $\ell_{-1}, \ell_0, \ell_1$, so this is a crude estimate. Alternatively, it is easily seen that the dimension of the characteristic variety of I_n is $(2n)$, implying holonomy (see e.g. Ref. 31). Making use of conformal invariance, if f is annihilated by I_n, then one can write:

$$f(x_1, \dots, x_{2n}) = g([x_1, x_2, x_3, x_4], \dots, [x_1, x_2, x_3, x_{2n}])$$

where g is a function of the $(2n-3)$ variables $y_i = [x_1, x_2, x_3, x_{3+i}]$, where $[.,.,.,.]$ is the cross-ratio. Since $\mathcal{L}_4, \dots \mathcal{L}_{2n}$ annihilate f, g is annihilated by operators $\tilde{\mathcal{L}}_1, \dots \tilde{\mathcal{L}}_{2n-3}$ such that the only second order differential term featuring in $\tilde{\mathcal{L}}_i$ is $\partial^2/\partial y_i^2$. Reasoning as above, this implies that the rank of I_n is at most 2^{2n-3}. We expect the rank to be exactly C_n.

In the case $n = 2$, if a function f is annihilated by I_2, then by conformal invariance ($\ell_{-1} f = \ell_0 f = \ell_1 f = 0$), one can write

$$f(x_1, \dots, x_4) = g([x_1, \dots, x_4])$$

where $[., ., ., .]$ is the cross-ratio. Then f is annihilated by I_2 if and only if g satisfies the second order ODE:

$$3u(1-u)g''(u) + (2-4u)g'(u) = 0$$

which is a Fuchsian differential equation with three singular regular points on the Riemann sphere (namely $0, 1, \infty$), so it is essentially equivalent to a hypergeometric differential equation (see Ref. 31, p. 26). It is easily seen that the rank of I_2 is exactly 2; at a generic point, the solution space is generated by a constant function and by the function appearing in Cardy's formula. It is worth remarking that we could have chosen any cross-ratio (there are six of them) to get the same solution space, which translates into exceptional invariance properties of the considered ODE (under homographies that fixate the singular locus $\{0, 1, \infty\}$). In fact, this is the only second-order Fuchsian ODE (with singular locus $\{0, 1, \infty\}$) that is invariant under the substitutions $u \mapsto (1-u)$, $u \mapsto 1/u$, so that the three singular points play exactly symmetric roles.

3.2. Loop-Erased Random Walks and Fomin's Formulae

In Ref. 12, Fomin considers a loop-erased version of the well-known Karlin-McGregor Formula. Problems pertaining to the scaling limits of Fomin's formulae are studied in Ref. 15. From Wilson's algorithm,[30] we know that there is an exact identity between branches of a uniform spanning tree (UST) and loop-erased random walks (LERW). It turns out that the natural way to define non-intersecting LERWs is to consider disjoint branches of an ambient UST.

More precisely, consider the following situation. A simply connected domain D of the plane, with, say, smooth Jordan boundary, is approximated by a subgraph of a lattice with mesh ε. Then take a uniform spanning tree of this graph with wired boundary conditions. Consider now $x_1, \dots, x_n$ n points on the boundary and $y_1, \dots, y_n$ n points one lattice spacing away from the boundary, such that $x_1, \dots, x_n, y_n, \dots, y_1$ are in counterclockwise order. We condition on the event that the minimal subtree containing $y_1, \dots, y_n$ and the boundary has no triple point in the bulk, and that the branch of y_i connects to the boundary at x_i, $i = 1, \dots, n$. Then take the scaling limit of these discrete paths. We get n non-intersecting paths connecting x_i to y_i, $i = 1, \dots, n$; each of them has density w.r.t. chordal SLE_2.[20]

In the discrete setting, let H be the harmonic measure for simple random walk on the lattice killed when it reaches the boundary. Then the probability of the event considered above is given by Fomin's formula:

$$\det\left(H(y_i, \{x_j\})\right)_{1\leq i,j\leq n}.$$

As follows from,[15] in the scaling limit (say in the upper half-plane $\mathbb{H}$), the Girsanov density of the system of paths w.r.t. independent chordal $\mathrm{SLE}_2(x_i \to y_i)$ is given by the scaling limit of this determinant divided by the product of its diagonal terms:

$$\det\left(\frac{1}{(x_i - y_j)^2}\right)_{i,j} \prod_i (x_i - y_i)^2.$$

In the continuum limit, we will see later an extension of this situation to values of κ between 0 and 8/3. Note that this discrete construction is reversible (for slightly different boundary conditions). This situation further illustrates the close relationship between LERW and discrete harmonic measure and its application to scaling limits. In the next section, we shall make use of the connection between UST and reflected harmonic measure.

3.3. Uniform Spanning Trees

If we specialize the results we shall obtain later in this paper to $\kappa = 8$, the formulae become symmetric in the $(2n)$ variables and have a nice Riemann surface interpretation. On the other hand, chordal SLE_8 is known to be the scaling limit of (the Peano curve of) the Uniform Spanning Tree (see Ref. 20). It is not hard to think of boundary conditions that enable to define multiple exploration processes in the discrete setting. From Wilson's algorithm,[30] it is then possible to extract the drift terms. Though, the connection with our formulae is not quite immediate, so we shall give some details here.

Consider a discrete lattice approximation of a bounded simply connected with, say, piecewise smooth boundary and $(2n)$ marked points $x_1, \ldots, x_{2n}$ on the boundary. The n boundary arcs $(x_2, x_3), (x_4, x_5), \ldots, (x_{2n}, x_1)$ are wired, and connected by an external wiring; the n other boundary arcs are free (i.e. wired for the dual graph). One then samples uniformly from spanning trees with these conditions. If we erase the external wiring, one gets n trees (corresponding to the n wired boundary components). Each vertex z is connected to one of the wired boundary component with probability given by the (graph) harmonic measure, from Wilson's algorithm. In the scaling limit, this should converge to harmonic measure with normal reflection on the free parts (see Fig. 4).

With these conditions, the Peano path starting at x_{2i} ends at x_{2i+1}, so that the boundary conditions fix a pairing of the $(2n)$ points; note also that this pairing

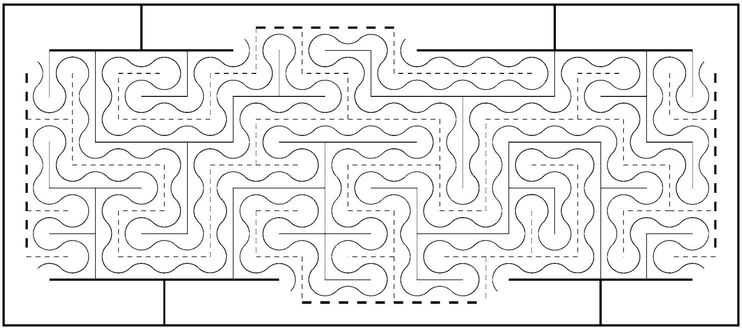

Fig. 4. Tree (solid), dual tree (dashed), Peano paths (curved).

is different from the one most natural for Fomin's formulae. Assume now for simplicity that the domain in the upper half-plane $\mathbb{H}$, and $x_1 < x_2 < \cdots < x_{2n}$. We want to prove that this situation corresponds in the scaling limit to SLE_8's whose drift terms are given by log-derivatives of the "partition function":

$$\psi(\mathbf{x}) = \prod_{1\leq i<j\leq 2n}(x_j - x_i)^{1/4}\int_C \prod_{1\leq i<j<n}(u_j - u_i)\prod_{i=1}^{n}\frac{du_i}{\left(\Pi_{j=1}^{2n}(u_i - x_j)\right)^{1/2}}$$

where C is a Cartesian product of $C_1, \ldots, C_{n-1}$, where C_i is a cycle circling clockwise round the segment (x_{2i-i}, x_{2i}). Note that one can choose a determination of the integrand on such cycles. In this expression, rewrite the product $\Pi_{i<j}(u_j - u_i)$ as a Vandermonde determinant. Define also:

$$\omega_i = \frac{u^{i-1}du}{\left(\Pi_{j=1}^{2n}(u - x_j)\right)^{1/2}},\quad \omega_i' = \frac{(u - x_1)^{i-1}du}{\left(\Pi_{j=1}^{2n}(u - x_j)\right)^{1/2}}$$

for $i \in \{1, \ldots n-1\}$. It is well-known that $(\omega_1, \ldots, \omega_{n-1})$ is a basis of abelian differentials of the first kind for the hyperelliptic curve of genus $g = n - 1$:

$$t^2 = (s - x_1)\cdots(s - x_{2n})$$

and $(C_1, \ldots, C_{n-1})$ is half of a canonical homology basis for this curve (see e.g. Ref. 11 for background on compact Riemann surfaces). Then ψ can be written as a determinant of periods:

$$\psi(\mathbf{x}) = \prod_{1\leq i<j\leq 2n}(x_j - x_i)^{1/4}\det\left(\int_{C_j}\omega_i\right) = \prod_{1\leq i<j\leq 2n}(x_j - x_i)^{1/4}\det\left(\int_{C_j}\omega_i'\right).$$

Consider the Peano exploration starting from x_1 and ending at x_{2n}. We want to recover the form of ψ from discrete arguments. First, we note that the situation can be seen as a degenerate case of a UST in a multiply-connected domain. Consider

$n-1$ compact holes $K_1, \dots \kappa_{n-1}$ in $\mathbb{H}$, with the following conditions: two points x_1 and x_{2n} are marked on $\mathbb{R}$; $(x_1 x_{2n})$ is free and $(x_{2n} x_1)$ is wired; the boundaries of the holes are wired, and all the wired parts are considered as wired together. One recovers the previous situation is the holes are segments close to $\mathbb{R}$. If z is any bulk point, then Wilson's algorithm provides $n-1$ martingales for the Peano exploration process. More precisely, let H_t be the remaining domain after time t of the exploration (for some time parameterization); and Harm is the harmonic measure in H_t with normal reflection on the free part of the boundary $((x_1 x_{2n}))$. Then:

$$t \mapsto M_t = {}^t(\mathrm{Harm}(z, \partial K_1), \dots \mathrm{Harm}\,(z, \partial K_{n-1}))$$

is a vector-valued martingale (see Ref. 20). Following Ref. 20, and the work of Makarov and Zhan for SLE in multiply connected domains (see Ref. 32), one determines from this the driving process of the scaling limit. We use here the notations and conventions of Ref. 8, So (g_t) is a family of conformal equivalences $H_t \to \mathbb{H}$ extending through the holes and with hydrodynamic normalization at infinity:

$$\partial_t g_t = \frac{2}{g_t - X_t}, \, dX_t = \sqrt{\kappa} dB_t + \kappa \frac{\partial_x \psi}{\psi} dt$$

for some covariant function ψ of the configuration, which we want to identify. Let z be a point on the free part of the boundary close to $x = x_1$; expanding at $z = x$, one gets:

$$M_t = (z-x)^{1/2} a_t + (z-x)^{3/2} b_t + \cdots$$

the half-integer exponents coming from the normal reflection. The martingale condition at $t = 0$ gives:

$$\left(\frac{\kappa}{2}\partial_{xx} + \kappa \frac{\partial_x \psi}{\psi}\partial_x + \frac{2}{z-x}\partial_z + \cdots\right)\left((z-x)^{1/2} a + (z-x)^{3/2} b + \cdots\right) = 0$$

where a and b are now (vector-valued) functions of the initial configuration. Considering the terms in $(z-x)^{-3/2}$ and $(z-x)^{-1/2}$ in this equation, one gets the necessary conditions $\kappa = 8$ and:

$$4\frac{\partial_x \psi}{\psi} a + 4\partial_x a = 6b.$$

So the drift term $\partial_x \log \psi$ can be recovered from a, b, that are coefficients in the expansion of the harmonic measure. Note that $\partial_x \log \psi$ is a real number, and the condition is a relation on vectors. Now we want to prove that ψ as defined above satisfies this equation in the degenerate case where $K_1 = (x_2 x_3), \dots K_{n-1} = (x_{2n-2} x_{2n-1})$. We assume that $n > 2$ (the case $n = 2$ being comparatively trivial).

Let P be the matrix:

$$P = \left(\int_{C_i} \omega'_j\right)_{1\le i,j<n}$$

and $Q(z)$ be the vector $\int_x^z {}^t(\omega'_1, \dots, \omega'_{n-1})$. Then it is easy to see that $\Re(P^{-1}Q)$ is the image of the harmonic measure vector H_0 by a fixed triangular matrix. In fact, the imaginary part will also give a martingale. So we can replace the harmonic measure vector by $P^{-1}Q$ in what follows, for simplicity. Hence, if $(e_1, \dots e_{n-1})$ is the standard basis of $\mathbb{C}^{n-1}$, one gets:

$$a = P^{-1}(2e_1)\prod_{i>1}(x_i - x)^{-1/2},\ b = P^{-1}\left(-\frac{1}{3}\left(\sum_{i>1}\frac{1}{x - x_i}\right)e_1 + \frac{2}{3}e_2\right)\prod_{i>1}(x_i - x)^{-1/2}$$

where $P^{-1}Q(z) = (z - x)^{1/2}a + (z - x)^{3/2}b + \cdots$. Using the fact that:

$$\partial_x \det(P) = \det(P)\mathrm{Tr}(P^{-1}\partial_x P)$$

and $\psi = \Pi_{1\le i<j}(x_j - x_i)^{1/4}\det(P)$, after simplifications, we have just to check that:

$$8\mathrm{Tr}(P^{-1}\partial_x P)e_1 + 8P\partial_x P^{-1}e_1 = 4e_2.$$

This follows from $P\partial_x(P^{-1}) + (\partial_x P)P^{-1} = 0$ and the fact that $\partial_x\omega'_{i+1} = (1/2 - i)\omega'_i$ if $i \in \{1, \dots, n-2\}$, so that:

$$\partial_x P = \begin{pmatrix} * & * & \cdots & \cdots & * \\ -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & \frac{5}{2} - n & 0 \end{pmatrix} P$$

and the previous identity can be read from the first column of $(\partial_x P)P^{-1}$.

3.4. Non-intersection and the Restriction Property

As pointed out in Ref. 8, the restriction property for $SLE_{8/3}$ can be put to use to get natural constructions of non-intersecting SLEs. For simplicity, consider first a simply-connected domain in $\mathbb{C}$ with four points marked on the boundary, say $(\mathbb{H}, x_1, x_2, x_3, x_4)$, where $\mathbb{H}$ is the upper half-plane. Consider two independent $SLE_{8/3}$'s in $\mathbb{H}$, from x_1 to x_2 and x_3 to x_4 resp., with traces γ, γ'. Say (g_t) is the family of conformal equivalences associated with the first one (time parameterization is unimportant here). The restriction property for the second SLE implies that the law of $g_t(\gamma')$ conditionally on $\{\gamma \cap \gamma' = \emptyset\}$ is the law of an $SLE_{8/3}$ from $g_t(x_3)$ to $g_t(x_4)$ conditioned not to intersect $g_t(\gamma_{(t,\infty)})$. Using at the same time the Markov property and the restriction property for the first SLE, one gets that $(g_t(\gamma), g_t(\gamma'))$ conditionally on $\{\gamma \cap \gamma' = \emptyset\}$ is distributed as $(\tilde{\gamma}, \tilde{\gamma}'))$ where $\tilde{\gamma}$ and $\tilde{\gamma}'$ are independent $\text{SLE}_{8/3}$'s going from $g_t(\gamma_t)$ to $g_t(x_2)$ and from $g_t(x_3)$ to $g_t(x_4)$ respectively, conditioned not to intersect. Indeed:

$$\mathcal{L}(g_t(\gamma_{(t,\infty)}), g_t(\gamma')|\gamma \cap \gamma' = \emptyset) = \mathcal{L}(g_t(\gamma_{(t,\infty)}), g_t(\gamma')|\gamma_{(0,t)} \cap \gamma' \\ = \emptyset, g_t(\gamma_{(t,\infty)}) \cap g_t(\gamma') = \emptyset) = \mathcal{L}(\tilde{\gamma}, \tilde{\gamma}'|\tilde{\gamma} \cap \tilde{\gamma}' = \emptyset)$$

Moreover, the system (γ, γ') has also a restriction property. This can be generalized to n SLEs connecting $(2n)$ points on the boundary, and conditoned on no intersection between any two of them. This induces a topological constraint on the order of the marked points $(x_1, \ldots, x_{2n})$ on the boundary. If the points $(x_1, \ldots, x_{2n})$ are in cyclical order on the boundary, one is interested in "non-crossing pairings" of these points (deciding that an SLE connects x_i and x_j pairs x_i and x_j). The number of such pairings is given by Catalan's number C_n. The value of the probability of non-intersection associated with such a pairing can be seen as a function on the moduli space $\mathcal{M}_{2n}$, and satisfies $(2n)$ evolution equations.[8]

If $\kappa \in (0, 8/3)$, this can be extended using loop-soups of intensity $\lambda = (6-\kappa)(8-3\kappa)/2\kappa$ (see Refs. 16, 21). More precisely, consider $(2n)$ marked points on the boundary, n independent SLE_κ connecting these points, and independent loop-soups $L_2, \ldots, L_n$ with intensity λ_κ. If the pairing is non-crossing, there is a positive probability that no loop in L_j intersects $\cup_{i<j}\gamma^i$ and γ^j, $j = 2, \ldots, n$. Conditioning on this event (and assuming reversibility), one gets a Markov property as above for each of the boundary points. Here one uses the Markov property of SLE, the restriction property for the loop-soup, the identity connecting loop-soup, SLE, and restriction measures,[16] and the Poissonian nature of the loop-soup. Note that one can rephrase the conditioning in a symmetric fashion. Indeed, if ν is the Brownian loop measure (denoted by μ^{loop} in Ref. 21), conditionally on the position of the SLEs, the probability that no loop in L_j intersects $\cup_{i<j}\gamma^i$ and γ^j, $j = 2, \ldots, n$ is:

$$\exp\left(-\lambda_\kappa \sum_j \nu(\{\delta : \delta \cap (\cup_{i<j}\gamma^i) \neq \emptyset, \delta \cap \gamma^j \neq \emptyset\})\right).$$

It readily appears that the sum in the exponent can be written as:

$$\sum_{n>0} \sum_{i_1<\cdots<i_n} (n-1)\nu(\{\delta : \delta \cap \gamma^j \neq \emptyset \ \text{ iff } \ j \in \{i_1, \ldots, i_n\}\})$$

by an inclusion-exclusion argument. So it is equivalent to condition on no loop in L_j intersecting (at least) j distinct SLEs, $j = 2, \ldots, n$.

Let us now formulate precisely and prove these results. Let $\kappa \in (0, 8/3]$. In the domain $(\mathbb{H}, x_1, y_1, \dots, x_n, y_n)$, consider n independent SLE$_\kappa$'s from x_i to y_i, $i = 1, \dots, n$, and independent loop-soups $L_2, \dots, L_n$ with intensity λ_κ. The SLEs are defined by conformal equivalences (g_t^i) and traces γ^i. Assuming that the pairing $(x_1, y_1), \dots, (x_n, y_n)$ is non-crossing, denote by $\mu_{(x_1,\dots,y_n)}$ the law of the SLEs conditionally on the event that no two SLEs intersect and no loop in L_j intersects γ^j and $\cup_{i<j}\gamma^i$ (if $\kappa < 8/3$, one can drop the first condition). A hull A is a compact subset of $\overline{\mathbb{H}}$ such that $\mathbb{H}\backslash A$ is simply connected, $A \cap \mathbb{R} \subset \overline{A \cap \mathbb{H}}$ and $x_1, \dots, y_n$ are not in A; ϕ_A is a conformal equivalence $\mathbb{H}\backslash A \to A$. Let L be yet another independent loop-soup with intensity λ_κ; the union of A and loops that intersect it is denoted by A^L; L^A is the collection of loops in L that do not intersect A.

Proposition 1. *Under the previous assumptions:*

(i) (Markov property) Under $\mu_{(x_1,\dots,y_n)}$, the law of $g_t^i(\gamma^1, \dots, \gamma^i_{[t,\infty)}, \dots \gamma^n)$ is that of $(\gamma^1, \dots, \gamma^n)$ under $\mu_{(g_t^i(x_1),\dots g_t^i(\gamma_t^i),\dots,g_t^i(y_n))}$ (up to time reparameterization).

(ii) Let $\psi(x_1, \dots, y_n)$ be the probability that no two SLEs intersect and no loop in L_j intersects γ^j and $(\cup_{i<j}\gamma^i)$, $j = 2, \dots, n$. Let $g_t = g_t^i$ for some $i \in \{1, \dots, n\}$. Then the process:

$$\psi(g_t(x_1), \dots, g_t(\gamma_t^i), \dots g_t(y_n)) \prod_{j\neq i} \left(g_t'(x_j) g_t'(y_j) \left(\frac{y_j - x_j}{g_t(y_j) - g_t(x_j)} \right)^2 \right)^{\alpha_\kappa}$$

is a martingale under the (unconditional chordal) SLE measure for the i-th SLE, where $\alpha_\kappa = (6-\kappa)/2\kappa$.

(iii) (Restriction property) Under $\mu_{(x_1,\dots,y_n)}$, the probability that A^L does not intersect $(\cup\gamma^i)$ is given by:

$$\frac{\psi(\phi(x_1), \dots, \phi(y_n))}{\psi(x_1, \dots, y_n)} \prod_i \left(\phi'(x_i)\phi'(y_i) \left(\frac{x_i - y_i}{\phi(x_i) - \phi(y_i)} \right)^2 \right)^{\alpha_\kappa}$$

where $\phi = \phi_A$. Moreover, the image under ϕ of $(\gamma_1, \dots, \gamma_n, L_2^A, \dots L_n^A, L^A)$ conditionally on "no loop in L_j connects γ_j to $\cup_{i<j}\gamma_i$, $j = 2, \dots, n$, and no SLE is connected to A by a loop in L" is distributed as $(\tilde\gamma_1, \dots, \tilde\gamma_n, \tilde L_2, \dots, \tilde L)$ where $\tilde\gamma^I$ are independent SLEs connecting $\phi(x_i)$ to $\phi(x_j)$, $\tilde L_2, \dots, \tilde L$ are independent loop-soups, conditionally on "no loop in $\tilde L_j$ connects $\tilde\gamma_j$ to $\cup_{i<j}\tilde\gamma_i$, $j = 2, \dots, n$."

Proof: If $n = 1$, (i) is the usual Markov property for SLE, (iii) is a result from, Ref. 16 and (ii) is an empty statement (since $\psi(x_1, y_1) = 1$).

By induction, assume that the results are proved for $(n-1)$ SLEs. Let us begin with (iii). Consider the event "no loop in L_j connects γ_j to $\cap_{i<j}\gamma_i$, $j=2,\ldots,n$, and no SLE is connected to A by a loop in L". The probability of this event, conditionally on the n independent SLEs is:

$$\exp\left(-\lambda_\kappa\left(\sum_{j\leq n}\nu(\{\delta:\gamma^j\cap\delta\neq\emptyset,(\cup_{i<j\gamma^i})\cap\delta\neq\emptyset\})+\nu(\{\delta:(\cup_j\gamma^j)\cap\delta\neq\emptyset,A\cap\delta\neq\emptyset\})\right)\right)$$

For illustration, let us momentarily assume that $n=2$. Then the sum in the exponential density reduces to:

$$\nu(\{\delta:\delta\cap\gamma^1\neq\emptyset,\delta\cap\gamma^2\neq\emptyset\})+\nu(\{\delta:\delta\cap(\gamma^1\cup\gamma^2)\neq\emptyset,\delta\cap A\cap\neq\emptyset\})$$

which we can rewrite by inclusion-exclusion as:

$$\begin{aligned}&\nu(\{\delta:\delta\cap\gamma^1\neq\emptyset,\delta\cap A\neq\emptyset\})+\nu(\{\delta:\delta\cap\gamma^2\neq\emptyset,\delta\cap A\neq\emptyset\})\\&\quad+\nu(\{\delta:\delta\cap\gamma^1\neq\emptyset,\delta\cap\gamma^2\neq\emptyset,\delta\cap A=\emptyset\}).\end{aligned}$$

The first term corresponds to the density of chordal SLE in $(\mathbb{H}\backslash A,x_1,y_1)$ w.r.t. chordal SLE in $(\mathbb{H},x_1,y_1)$; symmetrically, the second term is the density of chordal SLE in $(\mathbb{H}\backslash A,x_2,y_2)$ w.r.t. chordal SLE in $(\mathbb{H},x_1,y_1)$. This follows from the restriction property for a single SLE, applied twice. The remaining term is the mass of loops in $\mathbb{H}\backslash A$ that do not intersect the two SLEs. So (iii) follows when $n=2$. Let us get back to the general case: $n\geq 2$, and we assume that the assertions hold for $(n-1)$ SLEs.

The sum in the exponential can be written as:

$$\begin{aligned}&\sum_{j<n}\nu(\{\delta:\gamma^j\cap\delta\neq,(\cup_{i<j}\gamma^i)\cap\delta\neq\emptyset\})\\&\quad+\nu(\{\delta:\gamma^n\cap\delta\neq\emptyset,(\cup_{i<n}\gamma^i)\cap\delta\neq\emptyset,A\cap\delta=\emptyset\})\\&\quad+2\nu(\{\delta:\gamma^n\cap\delta\neq\emptyset,(\cup_{i<n}\gamma^i)\cap\delta\neq\emptyset,A\cap\delta\neq\emptyset\})\\&\quad+\nu(\{\delta:\gamma^n\cap\delta\neq\emptyset,(\cup_{i<n}\gamma^i)\cap\delta=\emptyset,A\cap\delta\neq\emptyset\})\\&\quad+\nu(\{\delta:\gamma^n\cap\delta=\emptyset,(\cup_{i<n}\gamma^i)\cap\delta\neq\emptyset,A\cap\delta\neq\emptyset\})\end{aligned}$$

and after rearranging:

$$\begin{aligned}&\sum_{j<n}\nu(\{\delta:\gamma^j\cap\delta\neq\emptyset,(\cup_{i<j}\gamma^i)\cap\delta\neq\emptyset\})+\nu(\{\delta:(\cup_{i<n}\gamma^i)\cap\delta\neq\emptyset,\\&\quad\times A\cap\delta\neq\emptyset\})+\nu(\{\delta:\gamma^n\cup\delta\neq\emptyset,A\cap\delta\neq\emptyset+\nu(\{\delta:\gamma^n\cap\delta\neq\emptyset,\\&\quad\times(\cup_{i<n}\gamma^i)\cap\delta\neq\emptyset,A\cap\delta=\emptyset\})\end{aligned}$$

The first two terms is treated by the induction hypothesis: it corresponds to the (unnormalized) density of $\mu_{(\mathbb{H}\backslash A,x_1,\ldots,y_{n-1})}$ w.r.t. $\mu_{(\mathbb{H},x_1,\ldots,y_{n-1})}$ The third term corresponds to the density of γ^n as an SLE in $(\mathbb{H}\backslash A, x_n, y_n)$ w.r.t. γ^n as an SLE in the full domain $(\mathbb{H}, x_n, y_n)$ (restriction property for a single SLE). The last term is the mass of loops in $\mathbb{H}\backslash A$ that intersect both γ_n and $\cup_{i<n}\gamma^i$ (from the restriction property of the loop-soup). So the exponential above is indeed the density of $\mu_{(\mathbb{H}\backslash A, x_1,\dots,y_n)}$ w.r.t. $\mu_{(\mathbb{H},x_1,\dots,y_n)}$. The statement on the joint law with loop-soups follows from the restriction property for the loop-soups $L_2, \dots, L$. (Note that we have exchanged loops between loop-soups, but these were only loops intersecting A).

The formula in (iii) is obtained by keeping track of masses of (unnormalized) measures in the induction scheme above; the new covariance factor in the product comes from the restriction property for a single SLE (say γ^n).

If A, B are two hulls and $A.B$ is the hull satisfying $\phi_{A.B} = \phi_B\phi_A$, then it is clear from the formula that the probability that $\cup_i\phi_A(\gamma^i)$ dsoes not touch B^L, conditionally on "no loop in L_j connects γ_j to $\cup_{i<j}\gamma_i$, $j = 2, \dots, n$, and no SLE is connected to A by a loop in L" multiplied by the probability that $\cup_i\phi_A(\gamma^i)$ does not touch A^L is the probability that $\cup_i\gamma^i$ does not touch $(A.B)^L$. This gives the (usual) restriction property for $(\cup_i\gamma^i)^L$, which is less precise than the statement (iii). (See also Ref. 8 for a discussion of these restriction measures).

Let us now prove (i),(ii). As explained above, the conditioning is in fact symmetric in the n SLEs. So, without loss of generality, we can assume that $i = n$. Reasoning as above, using the Markov property for γ^n and the restriction property (iii) for $(\gamma^1, \dots, \gamma^{n-1})$ (where A is replaced with $\gamma^n_{[0,t]}$), one gets easily (i) and (ii). □

When $\kappa = 2$, this situation connects with Fomin's determinantal formulae.[12,15] Indeed, adding the loops of a loop-soup with intensity λ_2 to an SLE_2, one gets a restriction measure with exponent 1, that has the same outer boundary as the Brownian Excursion. (In fact, the union of SLE_2 with the unfilled loops that intersect it has the same distribution as the range of the Brownian excursion, see Refs. 17, 32). For instance, consider the situation with two SLE_2's. Then the probability that no loop intersects the two SLEs is the same as the probability that the first SLE does not intersect the second one with loops attached, which is the same as the probability that an SLE_2 does not intersect a Brownian excursion (with respective endpoints x_1, x_2, x_3, x_4). This gives an interpretation of the symmetry of Fomin's formulae in the scaling limit (this symmetry follows from Wilson's algorithm in the discrete case).

Under mild regularity assumptions, Itô's formula and (ii) imply that ψ, which is conformally invariant by construction, is annihilated by the n operators:

$$\frac{\kappa}{2}\partial^2_{x_j} + \sum_{k\neq j}\frac{2\partial_{x_k}}{x_k - x_j} + \sum_k \frac{2\partial_{y_k}}{y_k - x_j} + \frac{2\partial_{y_j}}{y_j - x_j}$$

$$+\frac{\kappa-6}{\kappa}\sum_{k\neq j}\left(\frac{1}{x_k-x_j}-\frac{1}{y_k-x_j}\right)^2.$$

Under the assumption of reversibility (see Ref. 23), ψ is also annihilated by the n operators obtained by swapping the x and y variables. Note that the pairing is materialized in the constant terms of these operators. In fact, one can symmetrize the equations by a conjugation. Indeed, denoting $x_{n+j} = y_j$, the function

$$\psi(x_1,\ldots,x_{2n})\prod_j (x_{n+j}-x_j)^{1-6/\kappa}$$

is annihilated by the operators

$$\begin{cases} \frac{\kappa}{2}\partial_{kk}+\sum_{l\neq k}\frac{2\partial_l}{x_l-x_k}+\frac{\kappa-6}{\kappa}\sum_{l\neq k}\frac{1}{(x_l-x_k)^2}, & k=1,\ldots,2n \\ \sum_k \partial_k \\ \sum_k x_k\partial_k - n(1-6/\kappa) \\ \sum_k x_k^2\partial_k-(1-6/\kappa)(x_1+\cdots+x_{2n}) \end{cases}$$

the three first-order operators representing conformal covariance. This is the system (2.1). So under regularity and reversibility assumptions, each non-crossing pairing of the $(2n)$ points $(x_1,\ldots,x_{2n})$ gives rise to a solution of this system.

In the case where $8/3 < \kappa < 4$, one can expect that one can still define a natural law on non-intersecting SLEs, whose density w.r.t. independent chordal SLEs is given by $\exp(-\lambda_\kappa \nu(\ldots))$; this density is now unbounded:

$$d\mu_{(x_1,\ldots,y_n)}(\gamma_1,\ldots,\gamma_n)=\psi(x_1,\ldots,y_n)^{-1}\mathbf{1}_{\{\gamma_i\cap\gamma_j=\varnothing,i<j\}}\exp$$
$$\times\left(-\lambda_\kappa\sum_j\nu(\{\delta:\delta\cap(\cup_{i<j}\gamma^i)\neq\varnothing,\delta\cap\gamma^j\neq\varnothing\})\right)d\mu_{x_1,y_1}(\gamma_1)\ldots d\mu_{(x_n,y_n)}(\gamma_n)$$

where $d\mu_{x_i,y_i}$ is the measure induced on paths (say as elements of the Hausdorff space) by standard chordal SLE$_\kappa$. So ψ can be interpreted here as a (hopefully finite) partition function.

4. SOME PARTICULAR CASES

Let $\kappa > 0$ and $n \geq 1; x_1, \ldots, x_{2n}$, are boundary points of $\mathbb{H}$. Consider the operators (2.1):

$$\begin{cases} \frac{\kappa}{2}\partial_{kk} + \sum\limits_{l\neq k} \frac{2\partial_l}{x_l - x_k} + \frac{\kappa-6}{\kappa}\sum\limits_{l\neq k} \frac{1}{(x_l-x_k)^2}, & k = 1, \ldots, 2n \\ \sum_k \partial_k & \\ \sum_k x_k\partial_k - n(1-6/\kappa) & \\ \sum_k x_k^2\partial_k - (1-6/\kappa)(x_1+\cdots+x_{2n}) & \end{cases}$$

If φ is annihilated by these operators and ι is an involution of $\{1, \ldots, 2n\}$ with no fixed point (so that ι determines a pairing), denote:

$$\psi(x_1, \ldots, x_{2n}) = \varphi(x_1, \ldots, x_{2n}) \prod_{\{j,\iota(j)\}} (x_j - x_{\iota(j)})^{6/\kappa-1}.$$

Then ψ is conformally invariant (from the last three Eq. of (2.1)); moreover, consider an SLE_κ from x_j to $x_{\iota(j)}$, with associated conformal equivalences (g_t) (with hydrodynamic normalization) and driving process W_t. Then:

$$\psi(g_t(x_1), \ldots, W_t, \ldots, g_t(x_{2n})) \prod_{\{k,\iota(k)\}\neq\{j,\iota(j)\}} \left(g_t'(x_k)g_t'(x_{\iota(k)})\left(\frac{x_k - x_{\iota(k)}}{g_t(x_k) - g_t(x_{\iota(k)})}\right)^2\right)^{\alpha_\kappa}$$

is a local martingale (from the j-th Eq. of (2.1)).

We study this system in a few cases, before giving a set of formal solutions.

4.1. Case $n = 2$

If φ and ψ are as above (say for the pairing $\{(x_1, x_2), (x_3, x_4)\}$), then ψ is a conformally invariant function of four boundary points, so $\psi(x_1, \ldots, x_4) = f(r)$, where r is the cross-ratio:

$$r = \frac{(x_3 - x_2)(x_4 - x_1)}{(x_3 - x_1)(x_4 - x_2)}.$$

Now it is easy to check that φ is a solution of (2.1) iff $f(r)r^{-2\kappa}$ is a solution of the hypergeometric equation with parameters $(a, b, c) = (4/\kappa, 1 - 4/\kappa, 8/\kappa)$ (see Ref. 10).

Assume that $x_1, \ldots, x_4$ are in cyclical order. It is easy to see that the boundary conditions $\psi(x_1, x_2, x_3, x_4) = 0 (r = 0)$, $\psi(x_1, x_2, x_3, x_4) = 1 (r = 1)$ determine the following solution:

$$\psi(x_1, x_2, x_3, x_4) = \frac{\Gamma(4/\kappa)\Gamma(12/\kappa - 1)}{\Gamma(8/\kappa)\Gamma(8/\kappa - 1)} r^{2/\kappa} {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}, r\right)$$

for $\kappa < 8$. For $\kappa = 6$, one recovers Cardy's formula. For $\kappa \in (0, 8/3]$, this can be interpreted as the probability that two SLE_κ's connecting x_1, x_2 and x_3, x_4 resp.

do not intersect and are not connected by a loop in an independent loop-soup with intensity λ_κ.

A solution of the hypergeometric equation with parameters $(a, b, c) = (4/\kappa, 1 - 4/\kappa, 8/\kappa)$ can be written as:

$$\int_{C'} u^{-4/k}(1-u)^{12/\kappa-2}(1-ux)^{-4/\kappa}du$$
$$= x^{1-8/\kappa}\int_C v^{-4/k}(v-x)^{12/\kappa-2}(v-1)^{-4/\kappa}dv$$

where C is a formal linear combination of "cycles," i.e. paths of integrations over which the integrand is single-valued and are either closed or start and end at singular points of the integrand, i.e. $(0, 1, x, \infty)$. In terms of the original variables, one gets the following expression (inverting the roles of x_3 and x_4):

$$\varphi(x_1, x_2, x_3, x_4) = \prod_{i<j<4}(x_j - x_i)^{2/\kappa}\prod_{i<4}(x_4 - x_i)^{1-6/\kappa}$$
$$\times \int_C \prod_{i<4}(v - x_i)^{-4/\kappa}(v - x_4)^{12/\kappa-2}dv \qquad (2)$$

Note that although (2.1) is symmetric in the $(2n)$ variables $x_1, \ldots, x_{2n}$, the integrand is symmetric only in the first three variables. In fact, one can interchange the role of x_4 and, say, x_3 by a homographic change of variables. It is possible, but apparently not very practical, to rewrite this integrand in a completely symmetric fashion.

4.2. Case $\kappa = 2$

If one specializes the above expression to $\kappa = 2$, one gets a rational function, since ${}_2F_1(2, -1, 4, r) = 1 - r/2$; note that $2r(1 - r/2) = 1 - (1 - r)^2$. The corresponding solution φ is:

$$\varphi(x_1, x_2, x_3, x_4) = \frac{1}{(x_1 - x_2)^2(x_3 - x_4)^2} - \frac{1}{(x_1 - x_3)^2(x_2 - x_4)^2}$$
$$= \det\begin{pmatrix} (x_1 - x_2)^{-2} & (x_1 - x_3)^{-2} \\ (x_4 - x_2)^{-2} & (x_4 - x_3)^{-2}\end{pmatrix}$$

For a general value of n, it is possible (if a bit tedious) to check that:

$$\varphi(x_1, \ldots, x_{2n}) = \det\left((x_i - x_{j+n})^{-2}\right)_{1\le i,j\le n}$$

gives a solution of (2.1), corresponding to the pairing $\{(x_1, x_{n+1}), \ldots, (x_n, x_{2n})\}$. Other pairings gives different solutions; there are non-trivial linear relations between those solutions. These solutions correspond to the scaling limit of Fomin's formulae.[12,15] More precisely, multiplying the determinant by the product of its diagonal terms, one gets an alternating sum of probabilities of a certain non-intersection event. It is interesting to observe that if (x_i, x_j) are paired, the limit of this probability as $x_i \to x_j$ is given by a lower dimensional determinant, as it should. Likewise, the decorrelation of SLEs living on different scales is obvious from these formulae. For the particular pairing $(x_1, x_{2n}), \dots, (x_n, x_{n+1})$, if the points are in cyclic order, the probability of the corresponding non-intersection event is given by a single determinant.

Each permutation of $x_1, \dots, x_{2n}$ in this formula give a solution of the system; there are (many) non-trivial linear relations between those solutions. In general, other (geometric) pairings do not correspond to a single determinant, but to a linear combination of determinants of this type. One has to determine the solution satisfying appropriate boundary conditions. It is clear that as $x_i \to x_j$, each determinant (divided by its diagonal terms) goes either to 0 or to a lesser order determinant not involving x_i, x_j.

4.3. Case $\kappa \to \infty$

Consider the following system:

$$\begin{cases} \partial_{kk}, & k = 1, \dots, 2n \\ \sum_k \partial_k & \\ \sum_k x_k \partial_k - n & \\ \sum_k x_k^2 \partial_k - (x_1 + \cdots + x_{2n}) & \end{cases}$$

which can be seen as the (somewhat degenerate) limit of (2.1) as κ goes to infinity. Then it is easy to see that solutions (functions annihilated by those operators) are polynomials in $x_1, \dots, x_{2n}$ homogeneous of degree n, of partial degree at most 1 in each variable. The vector space of these polynomials has dimension $C(2n, n)$; the dimension of the subspace of such polynomials that are also translation invariant is

$$C(2n, n) - C(2n, n-1) = C(2n, n)/(n+1) = C_n.$$

Considering the derivative of these polynomials with respect to a variable, and by induction on the number of variables, one gets that this linear space is spanned by:

$$(x_1 - x_{n+1}) \cdots (x_n - x_{2n})$$

and polynomials obtained by action of the symmetric group on this one. Note that these spanning polynomials also satisfy the last relation in the system. So the solution space has dimension exactly C_n.

4.4. Case $\kappa = 6, 2n = 6$

Here we consider the system satisfied by the crossing probabilities for critical percolation in an hexagon with alternate boundary conditions (or a simply connected domain with six points marked on the boundary). In this situation, there are $C_3 = 5$ elementary crossing events, with probabilities adding up to 1 (see Fig. 2). A solution φ of this system is conformally invariant (and not only covariant); so one can send x_1, x_2, x_6 to $\infty, 0, 1$ by an homography, and φ becomes a function of three real variables y_1, y_2, y_3. This solution must be annihilated by six second-order operators; it turns out that there is (exactly) one linear relation between these operators. Singularities for these equations occur when two of the y variables are equal, or one is equal to $0, 1, \infty$. Consider an open set U of regular points. Let $D(U)$ be the subalgebra of differential operators in $\mathbb{C}(y_1, y_2, y_3)[\partial_1, \partial_2, \partial_3]$ with coefficients regular in U, and I be the left ideal of differential operators generated by these equations.

It is possible to write the system as an integrable, rank 5 Pfaffian system (see e.g. Ref. 31; this is also true for any $\kappa > 0$). Also, note that the constant function is a solution; we will use this simple fact to explicitate solutions of the system. Consider the image of the the solution space in U (one may also think in terms of local systems, sheaves of algebras, ...). Its image under the operator ∂_1 has dimension 4. Let J be the left ideal:

$$J = \{L \in D(U) : L\partial_1 \in I\}.$$

Then J is a holonomic ideal of rank 4. There is a well-known example of a holonomic system of rank 4, with 3 variables, and the same singular locus as J, namely the system of Lauricella's F_D function, which is the natural generalization of Appell's F_1 (see e.g. Ref. 1 and 10).

Recall that the system F_D with parameters $a, \gamma, \beta_1, \dots \beta_m$ in the variables $u_1, \dots, u_m$, is given by the linear operators:

$$\sum_{j=1}^{m} u_j(1-u_i)\partial_{ij} + (\gamma - (a + \beta_i + 1)u_i)\partial_i - \sum_{j \neq i} \beta_i u_j \partial_j - a\beta_i$$

for $i = 1, \dots, m$. A solution then also satisfies the Euler-Darboux equations (if $\gamma - a - 1 \neq 0$):

$$(u_i - u_j)\partial_{ij} - (\beta_j \partial_i - \beta_j \partial_j)$$

for $1 \leq i < j \leq m$. It is known that this system is of rank $(m+1)$ and that an integral representation of solutions is given by:

$$\int_C t^{a-1}(1-t)^{\gamma-a-1} \prod_j (1-tu_j)^{-\beta_j} dt.$$

By elimination in $D(U)$, it is possible to exhibit elements in J that generate a rank 4 ideal, the solution space of which is a conjugate of that of a particular F_D system. More precisely, with the help of a formal computation software, one can prove that if φ is a solution of the original system, then:

$$\frac{(y_1(y_2-y_1)(y_3-y_1)(1-y_1))^{2/3}}{((1-y_2)(1-y_3)(y_3-y_2)y_2y_3)^{1/3}}\partial_1\varphi(y_1,y_2,y_3)$$

is a solution of the F_D system with parameters:

$$a=1/3, \gamma=2/3, \beta_1=-4/3, \beta_2=2/3, \beta_3=2/3.$$

Similarly, in order to get a solution symmetric in the y variables, one can do the same computation for the operator $(y_1\partial_1+y_2\partial_2+y_3\partial_3)$, which by conformal invariance corresponds to a perturbation of x_6. Then it turns out that:

$$\frac{((1-y_1(1-y_2)(1-y_3))^{2/3}}{(y_1y_2y_3(y_2-y_1)(y_3-y_1)(y_3-y_2))^{1/3}}(y_1\partial_1+y_2\partial_2+y_3\partial_3)\varphi(y_1,y_2,y_3)$$

is a solution of the F_D system with parameters:

$$a=1/3, \gamma=8/3, \beta_1=2/3, \beta_2=2/3, \beta_3=2/3.$$

solutions of this system are given by

$$\int_C u^{-2/3}(1-u)^{4/3}((1-uy_1)(1-uy_2)(1-uy_3))^{-2/3}du$$

where as before C is an adequate "cycle." To get to φ, we write:

$$\frac{d}{ds}\varphi(sy_1,sy_2,sy_3)=((y_1\partial_1+y_2\partial_2+y_3\partial_3)\varphi)(sy_1,sy_2,sy_3)$$

which after trivial manipulations leads to the following expression for $\varphi(x_1,\dots,x_6)$:

$$\prod_{i<j<6}(x_j-x_i)^{1/3}\prod_{i<6}(x_6-x_i)^0\int_C\prod_{\substack{i\in\{1,2\}\\ j<6}}(u_i-x_j)^{-2/3}$$
$$\times\prod_{i\in\{1,2\}}(u_i-x_6)^0(u_2-u_1)^{4/3}du_1du_2. \tag{3}$$

We end this section by quoting a particular result for configurations with symmetries. Consider the domains $(\mathbb{U}, 1, u, j, ju, j^2, j^2u)$ where $\mathbb{U}$ is the unit disk, $j=e^{2i\pi/3}$, and u is on the arc $(1,j)$. Restricting a solution φ to such domains, one gets a function g of a single variable u. Now, computing in $D(U)$ as above, it is possible to derive a third-order ODE satisfied by g (of course this ODE has no constant term). The fact that the rank goes down is linked to the particular symmetries of the configurations we study, and is easily interpreted in terms of percolation observables (different elementary events have the same probability because of the threefold rotational symmetry of the configuration). Making the cubic change of variables $v = u^3$ (that sends the singularities $1, j, j^2$ to 1), then the quadratic change $w = -(v-1)^2/4v$ (that sends the singularities $0, \infty$ to ∞), so that $w = \sin^2(3\theta/2)$ if $u = e^{i\theta}$, it turns out that $w \mapsto w^{1/2}g'(w)$ satisfies the (classical) hypergeometric equation with parameters $(a, b; c) = (5/6, 5/6; 7/6)$. Hence g' belongs to the vector space spanned by:

$$h_1(w) = w^{-1/2}{}_2F_1\left(\frac{5}{6}, \frac{5}{6}; \frac{3}{2}; 1-w\right), h_2(w) = w^{-1/2}(1-w)^{-1/2} \times {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; \frac{1}{2}; 1-w\right)$$

Let $g_i(w) = \int_w^1 h_i(s)ds$. The function g belongs to the vector space spanned by $(1, g_1, g_2)$. Let us work out boundary conditions for crossing probabilities. Consider:

$$g(u) = \varphi((\mathbb{U}, 1, u, \dots, j^2u)) = \mathbb{P}((1, u) \leftrightarrow (j, ju) \leftrightarrow (j^2, j^2u))$$

the probability that the three blue sides belong to the same blue cluster (the "Mercedes" configuration). Let

$$g_+(w) = g(u) + g(j/u), g_-(w) = g(j/u) - g(u)$$

for u in the arc$(1, e^{i\pi/3})$ (note that when u goes from 1 to j, w goes from 0 to 1 and then back to 0). For parity reasons, it is easy to see that one can write $g_+ = c_1g_1 + c_3, g_- = c_2g_2$. We need to determine the three constants c_1, c_2, c_3. They are fixed by the boundary conditions: $g_+(0^+) = g_-(0^+) = 1$, and $g_+(w) - g_-(w) = O(\sqrt{w}) = O(u-1)$ as $w \searrow 0$ (the exponent here is three times the half-plane one-arm exponent). From the Euler integral:

$$\begin{aligned}{}_3F_2(a_1, a_2, a_3; \rho_1, \rho_2; x) &= B(a_1, \rho_1 - a_1)^{-1}B(a_2, \rho_2 - a_2)^{-1} \\ &\times \int_{[0,1]^2} s^{a_1-1}(1-s)^{\rho_1-a_1-1}t^{a_2-1}(1-t)^{\rho_2-a_2-1}(1-stx)^{-a_3}dsdt\end{aligned}$$

one gets:

$$\begin{aligned}g_1(0) &= B(5/6, 2/3)^{-1}\int_0^1(1-s)^{-1/2}\int_0^1 t^{-1/6}(1-t)^{-1/3}(1-st)^{-5/6}dsdt \\ &= {}_3F_2(1, 5/6, 5/6; 3/2, 3/2; 1)B(1, 1/2) \\ g_2(0) &= B(1/3, 1/6)^{-1}\int_0^1(s(1-s))^{-1/2}\int_0^1 t^{-2/3}(1-t)^{-5/6}(1-ts)^{-1/3}dsdt \\ &= {}_3F_2(1/2, 1/3, 1/3; 1, 1/2; 1)B(1/2, 1/2)\end{aligned}$$

$$= {}_2F_1(1/3, 1/3; 1; 1)B(1/2, 1/2) = \frac{\Gamma(1/3)\Gamma(1/2)^2}{\Gamma(2/3)^2}.$$

For the condition $(c_1h_1 - c_2h_2)(w) = O(w^{-1/2})$, we need the analytic continuation formulae:

$$\begin{aligned}
w^{1/2}h_1(w) &= \frac{\Gamma(3/2)\Gamma(-1/6)}{\Gamma(2/3)^2}{}_2F_1\left(\frac{5}{6}, \frac{5}{6}; \frac{7}{6}; w\right) \\
&\quad + \frac{\Gamma(3/2)\Gamma(1/6)}{\Gamma(5/6)^2}w^{-1/6}{}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{6}; w\right) \\
w^{1/2}h_2(w) &= \frac{\Gamma(1/2)\Gamma(-1/6)}{\Gamma(1/6)^2}{}_2F_1\left(\frac{5}{6}, \frac{5}{6}; \frac{7}{6}; w\right) \\
&\quad + \frac{\Gamma(1/2)\Gamma(1/6)}{\Gamma(1/3)^2}w^{-1/6}{}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{6}; w\right).
\end{aligned}$$

Hence we can conclude that:

$$c_2 = \frac{\Gamma(2/3)^2}{\Gamma(1/3)\Gamma(1/2)^2} = \frac{\sqrt{3}}{2\pi^2}\Gamma(2/3)^3, c_1 = c_2\frac{2\Gamma(5/6)^2}{\Gamma(1/3)^2} = \left(\frac{\sqrt{3}}{2^{2/3}\pi}\right)^5 \Gamma(2/3)^9.$$

So the probability of a configuration where two (given) blue sides and two yellow sides are connected for a regular hexagon ($u = e^{i\pi/6}$, $w = 1$, see Fig. 5) is:

$$\frac{1-c_3}{3} = \frac{c_1g_1(0)}{3} = \frac{2}{3}\left(\frac{\sqrt{3}}{2^{2/3}\pi}\right)^5 \Gamma(2/3)^9{}_3F_2(1, 5/6, 5/6; 3/2, 3/2; 1).$$

5. FORMAL SOLUTION

In this section, we discuss a family of formal solutions for the system (2.1). These are integrals taken on a "cycle"; they define actual solutions under conditions of convergence and determination of the integrand. We shall discuss later the problem of identifying probabilistic solutions.

Consider the function:

$$\phi(\mathbf{x}, \mathbf{u}) = \prod_{\substack{1\leq i<n \\ 1\leq j<2n}} (u_i - x_j)^{-\frac{4}{\kappa}} \prod_{1\leq i<n} (u_i - x_{2n})^{\frac{12}{\kappa}-2} \prod_{1\leq i_1<i_2<n} (u_{i_2} - u_{i_1})^{\frac{8}{\kappa}}$$
$$\prod_{1\leq j_i<j_2<2n} (x_{j_2} - x_{j_1})^{\frac{2}{\kappa}} \prod_{1\leq j<2n} (x_{2n} - x_j)^{1-\frac{6}{\kappa}}$$

where $\mathbf{x} = (x_1, \ldots, x_{2n})$, $\mathbf{u} = (u_1, \ldots, u_{n-1})$. We want to prove that a function of the parameters $(x_1, \ldots, x_{2n})$ of the form:

$$\int_C \phi(\mathbf{x}, \mathbf{u}) d\mathbf{u}$$

defines a solution of (2.1) for appropriate cycles C (that may depend on the x parameters). Note that this covers the cases (4.2) and (4.3). The conformal covariance conditions are easy to check. Denote by $\mathcal{L}_k$ the differential operator:

$$\mathcal{L}_k = \frac{\kappa}{2}\partial_{kk} + \sum_{l \neq k} \frac{2\partial_l}{x_l - x_k} + \frac{\kappa - 6}{\kappa} \sum_{1 \neq k} \frac{1}{(x_l - x_k)^2}.$$

Then the following lemma holds.

Lemma 1.

(i) If $1 \leq k < 2n$*, then:*

$$\mathcal{L}_k \phi = -\sum_{1 \leq i < n} \frac{\partial}{\partial u_i} \left(\frac{2\phi}{u_i - x_k} \right)$$

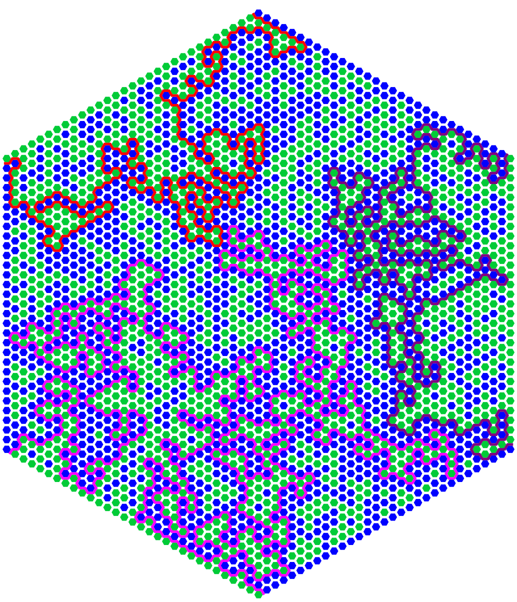

Fig. 5. Percolation in a regular hexagon.

(ii) *Moreover:*

$$\mathcal{L}_{2n}\phi = -\sum_{1\le i<n}\frac{\partial}{\partial u_i}\left(\left(\frac{2}{u_i-x_{2n}}+\frac{\kappa-8}{u_i-x_{2n}}\prod_{1\le j\le 2n}\frac{u_i-x_j}{x_{2n}-x_j}\right.\right.$$
$$\left.\left.\times\prod_{1\le j<n, j\neq i}\left(\frac{x_{2n}-u_j}{u_i-u_j}\right)^2\right)\phi\right)$$

Proof: (i) By symmetry, one can assume that $k=1$. Then:

$$\frac{\mathcal{L}_1\phi}{\phi}=\frac{\kappa}{2}\left(\left(-\frac{4}{\kappa}\sum_{1\le i<n}\frac{1}{x_1-u_i}+\frac{2}{\kappa}\sum_{1<j<2n}\frac{1}{x_1-x_j}+\left(1-\frac{6}{\kappa}\right)\frac{1}{x_1-x_{2n}}\right)^2\right.$$
$$\left.+\frac{4}{\kappa}\sum_{1\le i<n}\frac{1}{(x_1-u_i)^2}-\frac{2}{\kappa}\sum_{1<j<2n}\frac{1}{(x_1-x_j)^2}-\left(1-\frac{6}{\kappa}\right)\frac{1}{(x_1-x_{2n})^2}\right)$$
$$+\sum_{1<j<2n}\frac{2}{x_j-x_1}\left(-\frac{4}{\kappa}\sum_{1\le i<n}\frac{1}{x_j-u_i}+\frac{2}{\kappa}\sum_{\substack{1\le k<2n\\ k\neq j}}\frac{1}{x_j-x_k}+\left(1-\frac{6}{\kappa}\right)\frac{1}{x_j-x_{2n}}\right)$$
$$+\frac{2}{x_{2n}-x_1}\left(1-\frac{6}{\kappa}\right)\left(-2\sum_{1<i<n}\frac{1}{x_{2n}-u_i}+\sum_{1\le j<2_n}\frac{1}{x_{2n}-x_j}\right)$$
$$+\left(1-\frac{6}{\kappa}\right)\sum_{1<j\le 2n}\frac{1}{(x_j-x_1)^2}$$

which after simplifications leads to:

$$\frac{\mathcal{L}_1\phi}{\phi}=-2\left(1+\frac{4}{\kappa}\right)\sum_{1\le i<n}\frac{1}{(x_1-u_i)^2}-\frac{8}{\kappa}\sum_{\substack{1\le i<n\\ 1<j<2n}}\left(\frac{1}{x_1-u_i}\frac{1}{x_1-x_j}+\frac{1}{x_j-x_1}\frac{1}{x_j-u_i}\right)$$
$$-4\left(1+\frac{6}{\kappa}\right)\sum_{1\le i<n}\left(\frac{1}{x_1-x_{2n}}\frac{1}{x_1-u_i}+\frac{1}{x_{2n}-x_i}\frac{1}{x_{2n}-u_i}\right)$$
$$+\frac{16}{\kappa}\sum_{1\le i_1<i_2<n}\frac{1}{x_1-u_{i_1}}\frac{1}{x_1-u_{i_2}}$$

where we have used the identity:

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}=0$$

for $(a, b, c) = (x_1, x_i, x_j)$, $1 < i < j \le 2n$. Also:

$$\frac{1}{\phi}\frac{\partial}{\partial u_i}\left(\frac{2\phi}{u_i - x_1}\right) = \frac{2}{u_i - x_1}\left(-\left(1+\frac{4}{\kappa}\right)\frac{1}{u_i - x_1} - \frac{4}{\kappa}\sum_{1<j<2n}\frac{1}{u_i - x_j}\right.$$
$$\left. + \left(\frac{12}{\kappa} - 2\right)\frac{1}{u_i - x_{2n}} + \frac{8}{\kappa}\sum_{\substack{1\le i_2<n\\ i_2\ne i}}\frac{1}{u_i - u_{i_2}}\right)$$

Making use of the same identity as above, one can easily conclude.

(ii) The situation here is slightly more intricate than in (i). Let P be the rational function:

$$P = \frac{1}{\phi}\left(\mathcal{L}_{2n}\phi + \sum_{1\le i<n}\frac{\partial}{\partial u_i}\left(\frac{2\phi}{u_i - x_{2n}}\right)\right)$$

Reasoning as above, one gets the following identity:

$$P = \frac{(8-\kappa)(4-\kappa)}{\kappa}\left(\sum_{1\le i<n}\frac{3}{(u_i - x_{2n})^2} - 2\sum_{\substack{1\le i<n\\ 1\le j<2n}}\frac{1}{(u_i - x_{2n})(x_j - x_{2n})}\right.$$
$$\left. + \sum_{1\le j_1<j_2<2n}\frac{1}{(x_{j_1} - x_{2n})(x_{j_2} - x_{2n})} + 4\sum_{1\le i_1<i_2<n}\frac{1}{(u_{i_1} - x_{2n})(u_{i_2} - x_{2n})}\right).$$

On the other hand, if Q is the rational function:

$$Q = \frac{1}{\phi}\sum_{1\le i<n}\frac{\partial}{\partial u_i}\left(\left(\frac{8-\kappa}{u_i - x_{2n}}\prod_{1\le j<2n}\frac{u_i - x_j}{x_{2n} - x_j}\prod_{1\le j<n, j\ne i}\left(\frac{x_{2n} - u_j}{u_i - u_j}\right)^2\right)\phi\right)$$

then Q can be written as:

$$Q = \frac{(8-\kappa)(4-\kappa)}{\kappa}\sum_{1\le i<n}\frac{1}{u_i - x_{2n}}\prod_{1\le j<2n}\frac{u_i - x_j}{x_{2n} - x_j}\prod_{1\le j<n, j\ne i}\left(\frac{x_{2n} - u_j}{u_i - u_j}\right)^2$$
$$\times\left(-\sum_{1\le j<2n}\frac{1}{u_i - x_j} + \frac{3}{u_i - x_{2n}} + 2\sum_{1\le j<n, j\ne i}\frac{1}{u_i - u_j}\right).$$

So the statement reduces to the identity of rational functions $P = Q$. This is a bit tedious, and we include these computations for the sake of completeness. We can assume that $k \ne 4, 8$. Note also that P and Q are symmetric in the u variables.

Let us expand Q at $u_1 = x_{2n}$ (denote $\varepsilon = u_1 - x_{2n}$):

$$Q = \frac{(8-\kappa)(4-\kappa)}{\kappa}\frac{1}{\varepsilon}\left(1+\varepsilon\sum_{1\leq j<2n}\frac{1}{x_{2n}-x_j}+\varepsilon^2\sum_{1\leq j_1<j_2<2n}\frac{1}{(x_{2n}-x_{j_1})(x_{2n}-x_{j_2})}\right)$$
$$\times\left(1+\varepsilon\sum_{1<i<n}\frac{-2}{x_{2n}-u_i}+\varepsilon^2\left(\sum_{1<i_1<i_2<n}\frac{4}{(x_{2n}-u_{i_1})(x_{2n}-u_{i_2})}+\sum_{1<i<n}\frac{3}{(x_{2n}-u_i)^2}\right)\right)$$
$$\times\left(\frac{3}{\varepsilon}+2\sum_{1<i<n}\frac{1}{x_{2n}-u_i}-\sum_{1\leq j<2n}\frac{1}{x_{2n}-x_j}\right.$$
$$\left.+\varepsilon\left(\sum_{1\leq j<2n}\frac{3-1}{(x_{2n}-x_j)^2}-2\sum_{1<i<n}\frac{1}{(x_{2n}-u_i)^2}\right)+o(\varepsilon)\right.$$
$$=\frac{3}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\sum_{1\leq j<2n}\frac{3-1}{x_{2n}-x_j}+\sum_{1<i<n}\frac{2-6}{x_{2n}-u_i}\right)$$
$$+\sum_{1\leq j_1<j_2<2n}\frac{3-2}{(x_{2n}-x_{j_1})(x_{2n}-x_{j_2})}+\sum_{1<i_1<i_2<n}\frac{12-8}{(x_{2n}-u_{i_1})(x_{2n}-u_{i_2})}$$
$$+\sum_{1<i<n}\frac{9-2-4}{(x_{2n}-u_i)^2}+\sum_{1<j<2n}\frac{1-1}{(x_{2n}-x_j)^2}+\sum_{\substack{1<i<n\\1\leq j<2n}}\frac{-2+2-2}{(x_{2n}-x_j)(x_{2n}-u_i)}+o(\varepsilon).$$

From here, and using symmetry, it appears readily that:

$$P - Q = o((u_i - x_{2n}))$$

for $1 \leq i < n$. One can also check that Q is regular along $u_i = u_j$, for $1 \leq i < j < n$ (by symmetry it is enough to check that the term in $(u_i - u_j)^2$ vanishes). Considering $(P - Q)$ as a rational function of the u variables, it appears that it has no pole (even at infinity), so it is constant (i.e. a function only of the x variables). Setting $u_1 = x_{2n}$, it appears that $(P - Q)$ is identically zero, which concludes the proof. $\square$

One can rephrase the lemma as follows: there are differential $(n-2)$-forms $\omega_1, \ldots, \omega_{2n}$ with rational coefficients such that for $1 \leq k \leq 2n$:

$$\mathcal{L}_k(\phi du_1 \wedge \cdots \wedge du_{n-1}) = d(\phi\omega_k).$$

So for a "cycle" C, we get a (real) solution of (2.1) as soon as the following formal computation makes sense:

$$\mathcal{L}_k \int_C \phi du_1 \wedge \cdots \wedge du_{n-1} = \int_C \mathcal{L}_k(\phi du_1 \wedge \cdots \wedge du_{n-1})$$

$$= \int_C d(\phi\omega_k) = \int_{\partial C} \phi\omega_k = 0.$$

The cycle C must be such that ϕ has a single-valued determination on C, one can differentiate w.r.t. the x parameters in the integral, and $\phi\omega_k$ vanishes (to a sufficient order) on ∂C (which is true in particular if $\partial C = \emptyset$). In fact C is a cycle for the twisted de Rham homology associated with ϕ (see e.g. Ref. 24, Sec. 5.4). In the next section we give examples of such cycles, in relation with the probabilistic set-up.

6. EXPLICIT SOLUTIONS

In this section we propose choices of cycles of integration such that one gets well-defined solutions, and that these solutions can be interpreted in the probabilistic situations discussed earlier. We distinguish the cases $\kappa \in (0, 8/3]$ and $\kappa = 6$. Recall that (with an emphasis on the number of parameters):

$$\phi_n(\mathbf{x},\ \mathbf{u}) = \prod_{\substack{1<i<n \\ 1\leq j<2n}} (u_i - x_j)^{-\frac{4}{\kappa}} \prod_{1\leq i<n} (u_i - x_{2n})^{\frac{12}{\kappa}-2} \prod_{i\leq i_l<i_2<n} (u_{i_2} - u_{i_1})^{\frac{8}{\kappa}}$$
$$\prod_{1\leq j_1<j_2<2n} (x_{j_2} - x_{j_1})^{\frac{2}{\kappa}} \prod_{1\leq j<2n} (x_{2n} - x_j)^{1-\frac{6}{\kappa}}$$

where x $= (x_1, \ldots, x_{2n})$, $\mathbf{u} = (u_1, \ldots, u_{n-1})$.

Assume that $x_1 < \cdots < x_{2n}$ are boundary points of $\mathbb{H}$, and the involution ι with no fixed points define a non-crossing pairing of the $(2n)$ points $x_1, \ldots x_{2n}$. Recall that the fundamental group $\Pi_1(\mathbb{C}\backslash\{x_k, x_{\iota(k)}\}, z_0)$ is the free group generated by two elements $\sigma_k, \sigma_{\iota(k)}$ corresponding to loops around $x_k, x_{\iota(k)}$ respectively. A double contour loop around $x_k, x_{\iota(k)}$ corresponds to the commutator

$$\sigma_k\sigma_{\iota(k)}\sigma_k^{-1}\sigma_{\iota(k)}^{-1}$$

so that its image in $\Pi_1(\mathbb{C}\backslash\{x_k\}, z_o))$ and $\Pi_1(\mathbb{C}\backslash\{x_{\iota(k)}\}, z_o))$ is the identity. We assume that the other x points are outside this loop. Then ϕ_n is single-valued on such a double contour loop. Now let us enumerate the $(n-1)$ pairs that do not contain the last point x_{2n}:

$$\{\{x_k, x_{\iota(k)}\}, k = 1, \ldots, 2n\} = \{\{a_k, b_k\}, k = 1, \ldots (n-1)\} \cup \{\{x_{2n}, x_{\iota(2n)}\}\}.$$

Denote also $a_n = x_{\iota(2n)}$, $b_n = x_{2n}$, and assume that $a_k, < b_k$, $k = 1, \ldots n$. For $k = 1, \ldots, n-1$, let C_k be a double contour loop around a_k, b_k so that these loops do not intersect pairwise (see Fig. 6). Define C to be the Cartesian product of these loops:

$$C = C_1 \times \cdots \times C_{n-1}.$$

Of course, C is a function of $x_1, \dots, x_{2n}$. Then ϕ_n (as a function of $u_1, \dots, u_{n-1}$) is single-valued on C, and $\partial C = \emptyset$. Moreover, the integral of ϕ_n on C does not change if the loops are deformed.

If L is a double contour loop around 0,1, and p, q are numbers with real part larger than (-1), then:

$$\begin{aligned}\int_L t^p(1-t)^q dt &= \left(1 - e^{2i\pi q} + e^{2i\pi(p+q)}\right) - e^{2i\pi p}\int_0^1 t^p(1-t)^q dt\\ &= (1-e^{2i\pi p})(1-e^{2i\pi q})\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\end{aligned}$$

as is easily seen when L gets close to the unit segment. By analytic continuation in p and q, the second expression stays valid for general values of p and q. In the situation where $p = q$, one can use figure eight loops.

Assume that the two consecutive points x_k, x_{k+1}, $k+1 < 2n$, are paired and correspond to the loop C_j.
Then it is easy to see that:

$$\begin{aligned}&\lim_{x_{k+1}\to x_k}(x_{k+1}-x_k)^{6/\kappa-1}\int_c \phi_n(\mathbf{x},\mathbf{u})du_1\dots du_{n-1}\\ &= c\int_{\hat{C}}\phi_{n-1}(\hat{x},\hat{u})du_1\dots\widehat{du_j}\dots du_{n-1}\end{aligned}$$

where $\hat{\mathrm{x}} = (x_1, \dots, \widehat{x_k}, \widehat{x_{k+1}}, \dots, x_{2n})$, $\hat{\mathbf{u}} = (u_1, \dots, \widehat{u}_j, \dots, u_{n-1})$, and $\widehat{C} = C_1 \times \cdots \widehat{C_j} \times \cdots \times C_{n-1}$. The constant c is given by (up to a complex number of

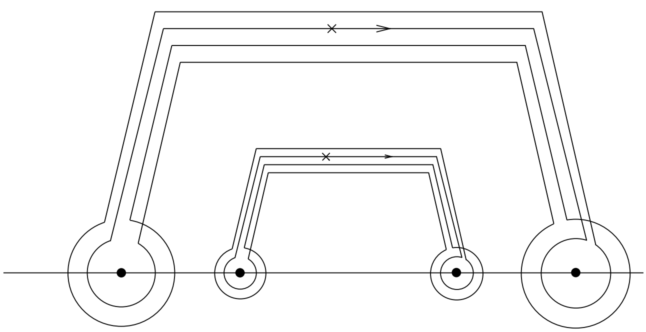

Fig. 6. Two non-intersecting double contour loops

modulus one depending on the choice of a determination for ϕ_n on C):

$$c = c_\kappa = 4\sin\left(\frac{4}{\kappa}\pi\right)^2 \frac{\Gamma\left(1-\frac{4}{\kappa}\right)^2}{\Gamma\left(2-\frac{8}{\kappa}\right)} = \frac{4\pi^2}{\Gamma\left(2-\frac{8}{\kappa}\right)\Gamma\left(\frac{4}{\kappa}\right)^2}$$

This constant is non zero, since we assume that $(8/\kappa) \notin \mathbb{N}$. If $n = 2$, then the limit is $c(x_4 - x_3)^{1-6/\kappa}$.

In this set-up, we can now formulate:

Proposition 3. *Let $\kappa \in (0, 8/3)$, $8/\kappa \notin \mathbb{N}$. Let $\gamma_1, \ldots, \gamma_n$ be (the traces of) n independent SLE$_\kappa$'s from a_k to b_k, $k = 1, \ldots n$, and $L_2, \ldots L_n$ be independent loop-soups with intensity λ_κ. Then:*

$$\mathbb{P}((\gamma_j)^{L_j} \cap (\cup_{i<j\gamma_i} = \emptyset), j = 2, \ldots, n = (c_\kappa)^{1-n} \prod_{k=1}^{n} (b_k - a_k)^{6/\kappa-1} \left| \int_C \phi_n(\mathbf{x}, \mathbf{u}) d\mathbf{u} \right|$$

Proof: Denote by $\tilde{\psi}(x_1, \ldots x_{2n})$ the right-hand side of the equation in the proposition. Then we want to prove $\tilde{\psi} = \psi$ (with the notations of Proposition 1). By construction, $\tilde{\psi}$ has the property (ii) of Proposition 1. Also, it is easy to see that $\tilde{\psi}$ is conformally invariant. So let us momentarily fix the values $x_1 = 0, x_{2n-1} = 1, x_{2n} = \infty$. Then $\tilde{\psi}$ extends continuously to the compactification of

$$\{(x_2, \ldots, x_{2n-2}) \in \mathbb{R}^{2n-3} \text{s.t. } 0 = x_1 < \cdots < x_{2n-1} = 1\}$$

so it is bounded. There is a $k \in \{1, \ldots, n-1\}$ such that a_k and b_k are two consecutive points on the real line. Consider the martingale associated with $\tilde{\psi}$ and the evolution of the k-th SLE. We use the hydrodynamic normalization at infinity; (gt) are the conformal equivalences associated with the SLE, which is defined up to time $\tau = \tau(b_k)$ (which is finite due to this arbitrary choice of normalization). Then g_τ defines a conformal equivalence between the unbounded connected component of $\mathbb{H}\backslash\gamma_k$ and $\mathbb{H}$. As $t \nearrow \tau$, W_t and $g_t(b_k)$ go to the same finite limit, and $g_t(a_j)$, $g_t(b_j)$ have finite (distinct) limits. So the martingale goes to:

$$\tilde{\psi}_{n-1}(g_\tau(\hat{\mathbf{x}})) \prod_{j\neq k} \left(g'_\tau(a_j) g'_\tau(b_j) \left(\frac{b_j - a_j}{g_\tau(b_j) - g_\tau(a_j)} \right)^2 \right)^{\alpha_\kappa}$$

with the obvious notations. The product can be written as:

$$\prod_{j\neq k} \mathbb{P}\left(\gamma_j^L \cup \gamma_k \neq \emptyset\right)$$

where L is an independent loop-soup. Now, as we remarked earlier, $\psi(\mathbf{x})$ does not depend on the ordering of the SLEs (this follows from the Poissonian nature of the loop soup). So we can assume that the $(n-1)$ SLEs loop-soups are reordered so that the last one corresponds to k (that is, k is re-indexed as n). As in Proposition 1, the restriction property for the loop-soup and for $(n-1)$ SLEs now implies that:

$$\psi(\mathbf{x}) = \mathbb{E}\Bigg(\mathbb{P}((\tilde{\gamma}_j)^{\tilde{L}_j} \cap (\cup_{i<j}\tilde{\gamma}_i) = \emptyset,$$

$$j = 2\dots(n-2)|g_\tau(a_j), g_\tau(b_j), j < n)\prod_{j<n} 1_{(\gamma_j)^{L'_j}\cap\gamma_n=\emptyset}\Bigg)$$

where $\tilde{\gamma}_j$ is an SLE from $g_\tau(a_j)$ to $g_\tau(b_j)$, and $\tilde{L}_j$, L'_j are independent loop-soups. Comparing with the limiting value of the martingale as $t \nearrow \tau$, one concludes by induction on n, using the optional stopping theorem. $\square$

Note that it appears to be difficult to determine analytically the boundary behaviour of $\tilde{\psi}$ along all boundary components. The point here is that the dynamics of the SLEs lead to an exit on a fixed boundary component; and commutation allows us to consider the SLEs in an appropriate order.

We gave probabilistic interpretations in the case $\kappa < 8/3$, $\kappa = 6$, $\kappa = 8$. We have identified corresponding integration cycles when $\kappa \leq 8/3$, $8/\kappa \neq \mathbb{N}$ and $\kappa = 8$. Let us discuss the remaining cases $\kappa = 8/m$, $m \geq 3$, and $\kappa = 6$.

The problem is to identify a cycle of integration corresponding to a geometric configuration of SLEs. In particular, one can consider the case of n nested paths connecting x_i to x_{2n+1-i}, $i = 1, \dots, n$, where $x_1, \dots, x_{2n}$ are in cyclical order on the boundary. This is the natural situation for Fomin's formulae (but not for the UST e.g.). Then the following cycle can be used (say when $\kappa < 4$):

$$C = C_1 \times \cdots \times C_{n-1}$$

where C_i is a loop starting and ending at x_{2n}, circling counterclockwise around x_{2n-i}, and leaving the other marked points on its right-hand side; moreover, the C_i''s are chosen so that they do not intersect except at x_{2n}, consequently the integrand has a single-valued determination on C.

In the case where $\kappa = 2$, induction on n and a residue computation give the determinant obtained as the scaling limit of Fomin's formula. One could also have chose $C' = C_1' \times \cdots \times C_{n-1}'$, with C_i a loop starting at x_1 and circling around x_{1+i} (and the special point in the integrand is now x_1, not x_{2n}); the solution is then proportional to the one corresponding to C. Here, using induction on n (and analytic continuation in κ), we can show that these two solutions are indeed proportional and have the correct asymptotic behaviour when two consecutive marked points collapse. Finding a cycle with geometrically prescribed boundary conditions (in the Weyl chamber $\{x_1 < \cdots < x_{2n}\}$) seems to be technically difficult in general.

Also, we remark here that the integral representation becomes somewhat degenerate in the particularly interesting case $\kappa = 6$. As in the case of standard hypergeometric functions, analytic continuation in the parameters (here κ) can be put to good use. Since the functions ψ can in this instance be interpreted as (scaling limits of) crossing probabilities, it would be nice to find real integration cycles (formal sums of Cartesian products of segments $[x_i, x_{i+1}]$) corresponding to these events (i.e. satisfying the right boundary conditions; in general, when two consecutive points collapse, the boundary condition is either 0 or a crossing probability for a configuration with $2(n-1)$ marked points). For small values of n (e.g. $2n = 6$), one can work out such cycles, though the general pattern is not very clear. Again for small n, these cycles give enough solutions, so the results of this paper can be seen as giving an algorithmic solution to the problem of computing crossing probabilities for the scaling limit of percolation with alternate boundary conditions.

As for general properties of the holonomic system we studied, the main question is to prove that the rank is indeed C_n From the case $\kappa \to \infty$, we see that the rank should be at most C_n. Under smoothness assumptions, the restriction construction give C_n distinct solutions when $0 < \kappa \leq 8/3$; this should be enough to prove that the rank is C_n in general (since all these systems then have a rank C_n Pfaffian form that should have an analytic continuation in κ).

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REFERENCES

1. P. Appell and J. Kampé de Fériet. *Fonctions hypergéometriques et hypersphériques. Polynômes d'Hermite*. (Gauthier-Villars, Paris, 1926).
2. F. Camia and C. M. Newman. The full scaling limit of two-dimensional critical percolation. *preprint, arXiv:math.PR/0504036.*
3. J. L. Cardy. Critical percolation in finite geometries. *J. Phys. A* **25**(4):L201–L206 (1992).
4. J. L. Cardy. Conformal invariance and percolation. *preprint, arXiv:math-ph/0103018* (2001).
5. J. L. Cardy. Crossing formulae for critical percolation in an annulus. *J. Phys. A* **35**(41):L565–L572 (2002).
6. P. Di Francesco, P. Mathieu and D. Sénéchal. *Conformal field theory. Graduate Texts in Contemporary Physics*. (Springer-Verlag, New York, 1997).
7. V. S. Dotsenko and V. A. Fateev. Conformal algebra and multipoint correlation functions in 2D statistical models. *Nuclear Phys. B* **240**(3):312–348 (1984).
8. J. Dubédat. Commutation relations for SLE. *preprint, arXiv:math.PR/0411299* (2004).
9. J. Dubédat. Excursion decompositions for SLE and Watts' crossing formula. *Probab. Theory Related Fields, to appear, http://dx.doi.org/10.1007/s00440-005-0446-3*, (2004).
10. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Higher transcendental functions. Vol. I.* Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman.

11. H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics.* 2nd edn. (Springer-Verlag, New York, 1992).
12. S. Fomin. Loop-erased walks and total positivity. *Trans. Amer. Math. Soc.* **353**(9):3563–3583 (electronic) (2001).
13. R. Friedrich and J. Kalkkinen. On conformal field theory and stochastic Loewner evolution. *Nuclear Phys. B* **687**(3):279–302 (2004).
14. G. Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften.* 2nd edn. (Springer-Verlag, Berlin, 1999).
15. M. J. Kozdron and G. F. Lawler. Estimates of random walk exit probabilities and application to loop-erased random walk. *Electron. J. Probab.* **10**:1442–1467 (electronic) (2005).
16. G. Lawler, O. Schramm and W. Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.* **16**(4):917–955 (electronic) (2003).
17. G. F. Lawler. The Laplacian-b random walk and the Schramm-Loewner evolution. *Illinois J. Math., to appear*.
18. G. F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs.* (American Mathematical Society, Providence, RI, 2005).
19. G. F. Lawler, O. Schramm and W. Werner. On the scaling limit of planar self-avoiding walk. In *Fractal geometry and application. A jubilee of Benoit Mandelbrot*, AMS Proc. Symp. Pure Math. 2002.
20. G. F. Lawler, O. Schramm and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32**(1B):939–995 (2004).
21. G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields* **128**(4):565–588 (2004).
22. H. T. Pinson. Critical percolation on the torus. *J. Statist. Phys.* **75**(5–6):1167–1177 (1994).
23. S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math. (2)* **161**(2):883–924 (2005).
24. M. Saito, B. Sturmfels, and N. Takayama. *Gröbner deformations of hypergeometric differential equations*, volume 6 of *Algorithms and Computation in Mathematics.* (Springer-Verlag, Berlin, 2000).
25. O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118**:221–288 (2000).
26. S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333**(3):239–244 (2001).
27. G. M. T. Watts. A crossing probability for critical percolation in two dimensions. *J. Phys. A* **29**(14):L363–L368 (1996).
28. W. Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pp. 107–195. Springer, Berlin, 2004.
29. W. Werner. Conformal restriction and related questions. *Probab. Surv.* **2**:145–190 (electronic) (2005).
30. D. B. Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, pp. 296–303, New York, 1996. ACM.
31. M. Yoshida. *Fuchsian differential equations.* Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, Braunschweig, 1987. With special emphasis on the Gauss-Schwarz theory.
32. D. Zhan. Random Loewner chains in Riemann surfaces. *PhD thesis* (2004).